Beyond the fundamental group

P. P. Boalch

Prelude / initial impetus

Cecotti-Vafa "Classification of $N=2$ supersymmetric theories" (1993)

Vacua $\{1, \ldots, n\}$

$s_{ij} \in \mathbb{Z}$ count solitons/BPS states $i \leftrightarrow j$

Braid group action $B_n \circlearrowleft \{s_{ij}\}$

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ \sigma_i \sigma_j = \sigma_j \sigma_i \ \text{if } |i-j| > 1 \rangle$$

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$s_{ij} \in \mathbb{Z}$ count solitons/BPS states $i \leftrightarrow j$

Braid group action $B_n \circlearrowleft \{ s_{ij} \}$

"Stokes matrix" $S = \begin{pmatrix} 1 & \ddots & s_{ij} \\ 0 & & \ddots \; 1 \end{pmatrix}$ $\qquad (e_i, e_j) = B_{ij}$

Cartan matrix $B = S + S^T \Rightarrow$ bilinear form $(\; , \;)$

Reflections r_i $\qquad r_i(v) = v - (v, e_i) e_i$

$$r_n r_{n-1} \cdots r_1 = -S^{-T} S \qquad \text{(Killing-Coxeter)}$$

$$\sigma_i(r_n, \dots, r_1) = (r_n, \dots, \overset{i+1}{r_i}, r_i r_{i+1} r_i, \dots, r_1) \qquad \text{(Hurwitz)}$$

$$\sigma_i(e_n, \dots, e_1) = (e_n, \dots, e_i, r_i(e_{i+1}), \dots, e_1)$$

$\Rightarrow$ action on S $\qquad (s_{ij} = (e_i, e_j))$

Prelude / initial impetus

Cecotti-Vafa "Classification of $N=2$ supersymmetric theories" (1993)

Dubrovin (1995) defined Frobenius manifolds ~ solutions to WDVV

- classified by Stokes matrix S modulo B_n action

$s_{ij} \in \mathbb{C}$ now, $S \in U_+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$

$\exists$ nonlinear B_n invariant Poisson structure on U_+

e.g $n=3$ $S = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$

$\{x, y\} = xy - 2z$ (& cyclic perm^{n}s)

$\sigma_1(x, y, z) = (-x, z - xy, y)$

$\sigma_2(\text{—}) = (y - xz, x, -z)$

preserves surfaces $x^2 + y^2 + z^2 - xyz = b$ (sympl. leaves)

Formulae for Poisson str. extended $\forall n$ by Ugaglia 1999 (using Korotkin-Samtleben)

Prelude / initial impetus

Cecotti-Vafa "Classification of $N=2$ supersymmetric theories" (1993)

Dubrovin (1995) defined Frobenius manifolds ~ solutions to WDVV

Transcendental characterisation (of this alg. Poisson str.)

$$D = \frac{d}{dz} - \left(\frac{U}{z^2} + \frac{V}{z}\right)dz \xrightarrow[\text{data at } 0]{\text{take Stokes}} S \in U_+$$

$$\begin{cases} U = \begin{pmatrix} u_1 & & 0 \\ & \ddots & \\ 0 & & u_n \end{pmatrix} & u_i \neq u_j \quad (\text{"canonical coords" on Frob mfd}) \\ V^T = -V, & \sim \quad V \in \mathrm{Lie}(SO_n(\mathbb{C}))^* \end{cases}$$

$$\dim \frac{n(n-1)}{2} = \dim U_+$$

This map $\mathrm{Lie}(SO_n(\mathbb{C}))^* \longrightarrow U_+$ is Poisson $\forall\ U \in \mathbb{C}^n \setminus \text{diags}$

- braiding of S ~ isomonodromic deformations of D (as U moves)

  ~ monodromy of nonlinear dif. eqn $d_U V = [V, \mathrm{ad}_U^{-1}[dU, V]]$

Prelude / initial impetus

Morals

① Jumps in BPS states ⟺ that of Stokes data under isomonodromy defm^n s.

② Braiding not just from motion of points on Riemann surfaces
- Can move "irregular type" U
- lots of generalisations

③ Seem to get some cool algebraic Poisson structures on spaces of Stokes data

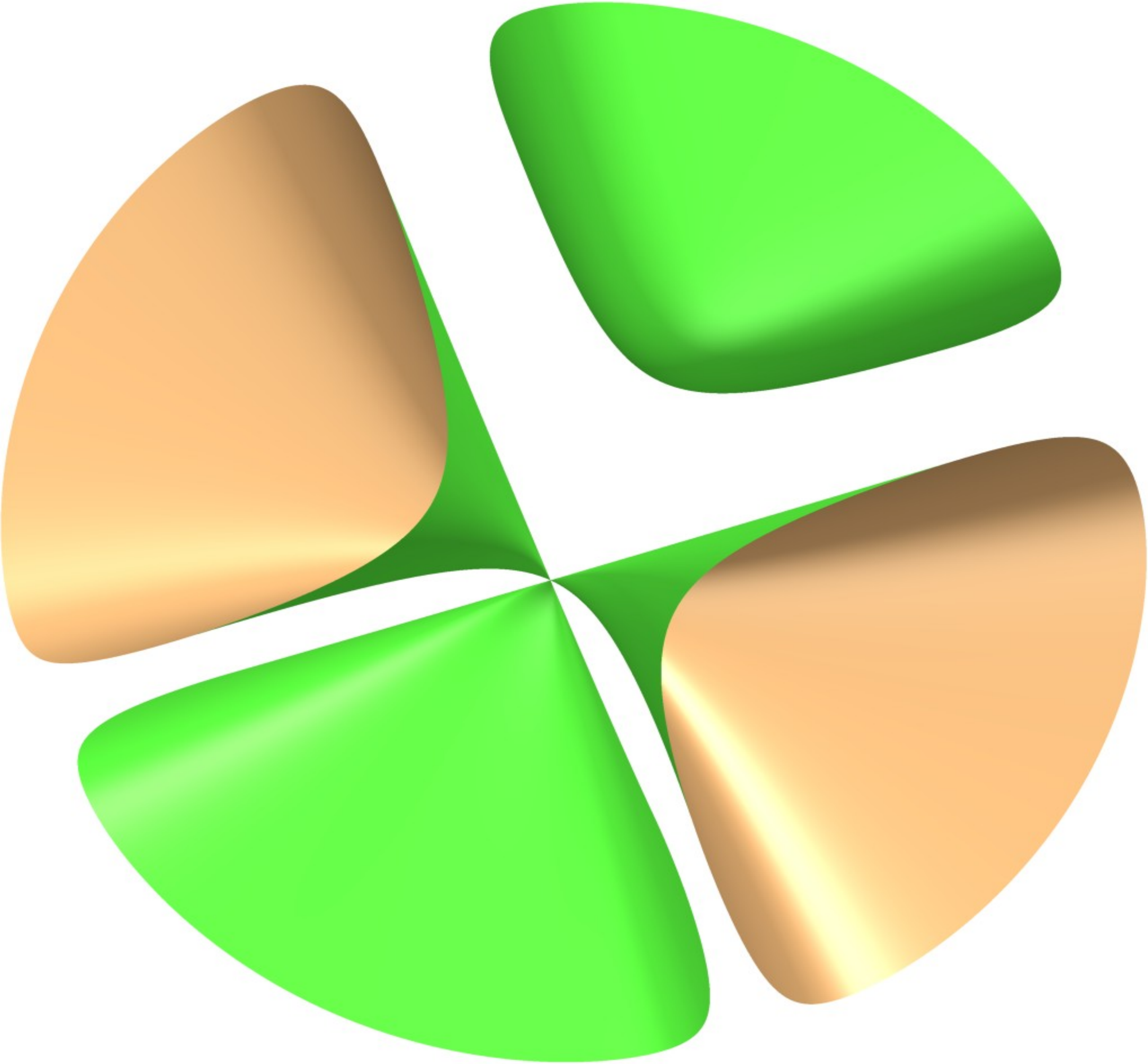
Exercise (cleaner example)

arbitrary $\in \mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{gl}_n(\mathbb{C})^*$

$$D = \frac{d}{dz} - \left(\frac{U}{z^2} + \frac{V}{z}\right)dz \xrightarrow{\text{Stokes}} U_+ \times T \times U_-$$

what is B_n invt Poisson str. here ?

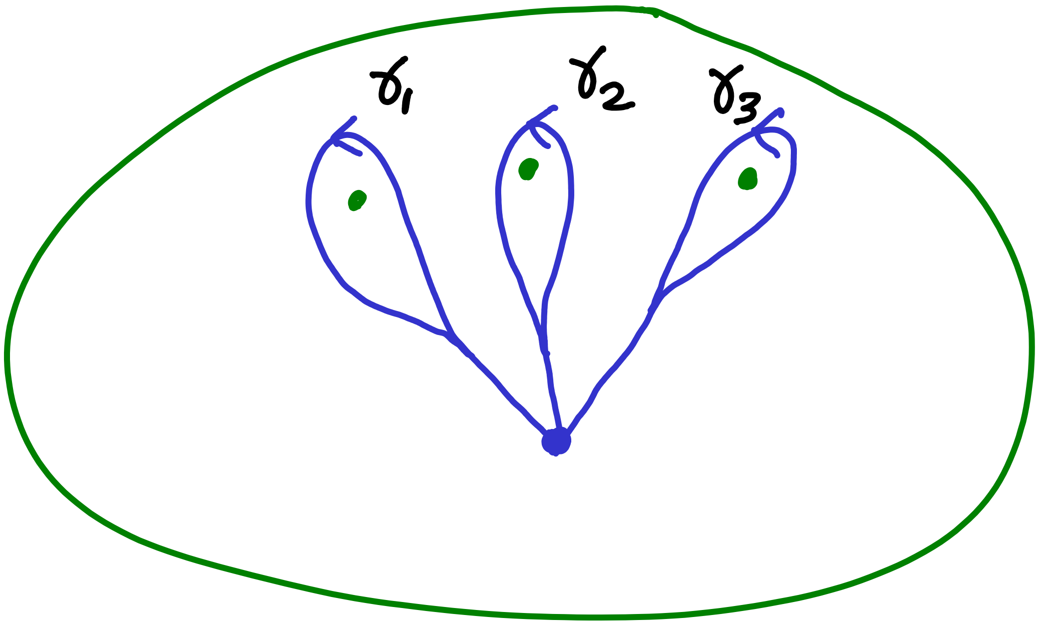
$$xyz + x^2 + y^2 + z^2 = b + b_1 x + b_2 y + b_3 z$$



Beyond the fundamental group

P. P. Boalch

E.g. $X = \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ (m-punctured two sphere)

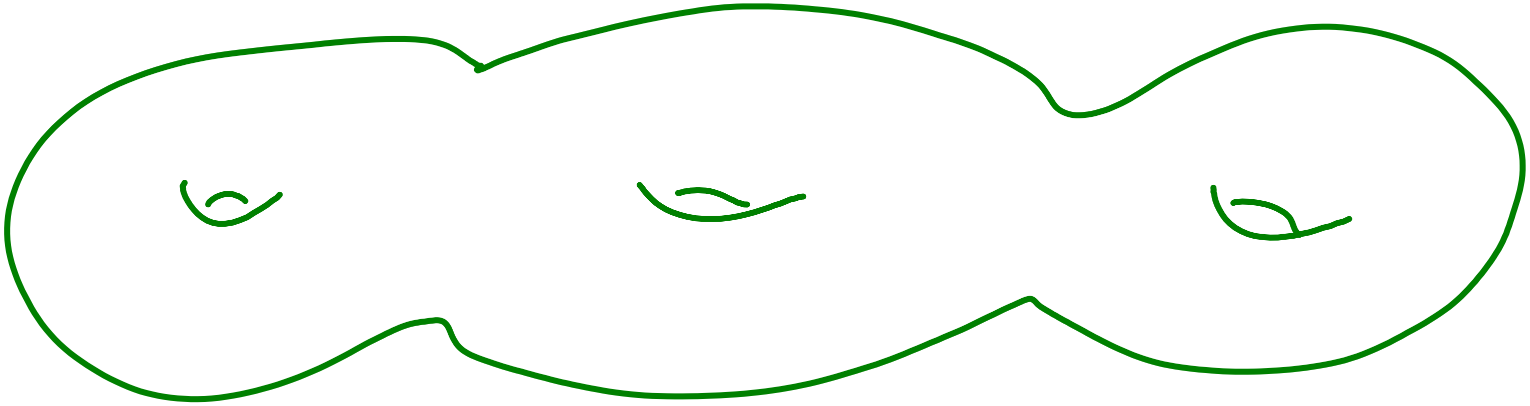


$$\pi_1(X, x) \cong \langle \gamma_1, \dots, \gamma_m \mid \gamma_1 \circ \cdots \circ \gamma_m = 1 \rangle$$

$$\cong \text{Free}_{m-1} \quad (\text{Free group})$$

m	0	1	2	3	4	5
π_1	1	1	$\mathbb{Z}$	Free_2	......	

E.g. X = genus g compact Riemann surface

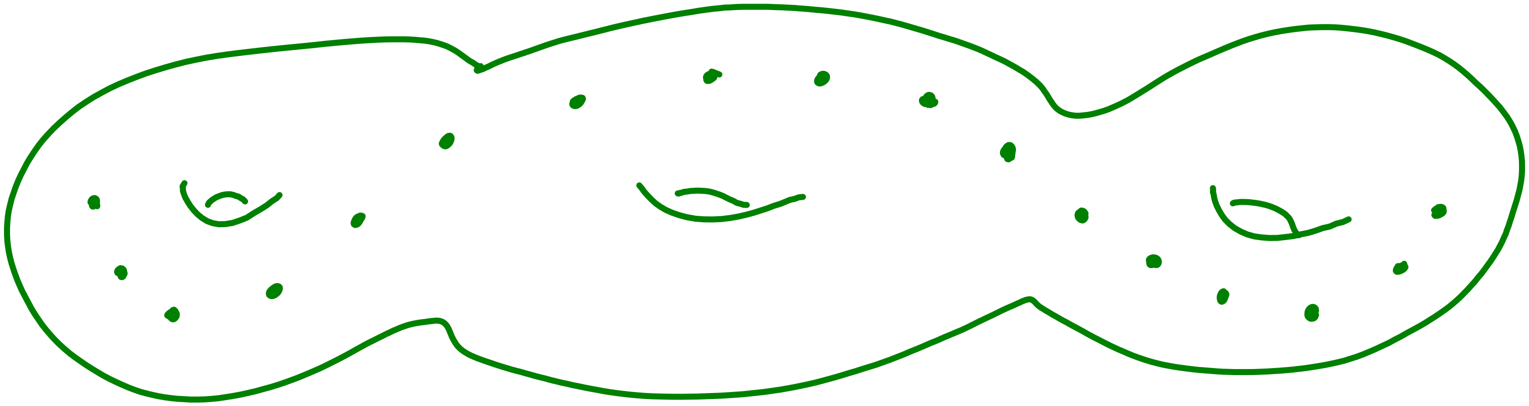


$$\pi_1(X) \cong \langle \alpha_1, \cdots, \alpha_g, \beta_1, \ldots, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$

E.g. $\pi_1 \cong \mathbb{Z}^2$ if $g=1$

E.g. X = m-punctured genus g compact Riemann surface



$$\pi_1(X) \cong \left\langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_m \;\middle|\; \prod_{1}^{g} [\alpha_i, \beta_i] \prod_{1}^{m} \gamma_j = 1 \right\rangle$$

"surface groups"

Nonabelian representations of surface groups arose in Riemann's work on the Gauss hypergeometric equation

$$z(1-z)\,y'' + (az+b)\,y' + c\,y = 0$$

[a, b, c constants, $y(z)$]

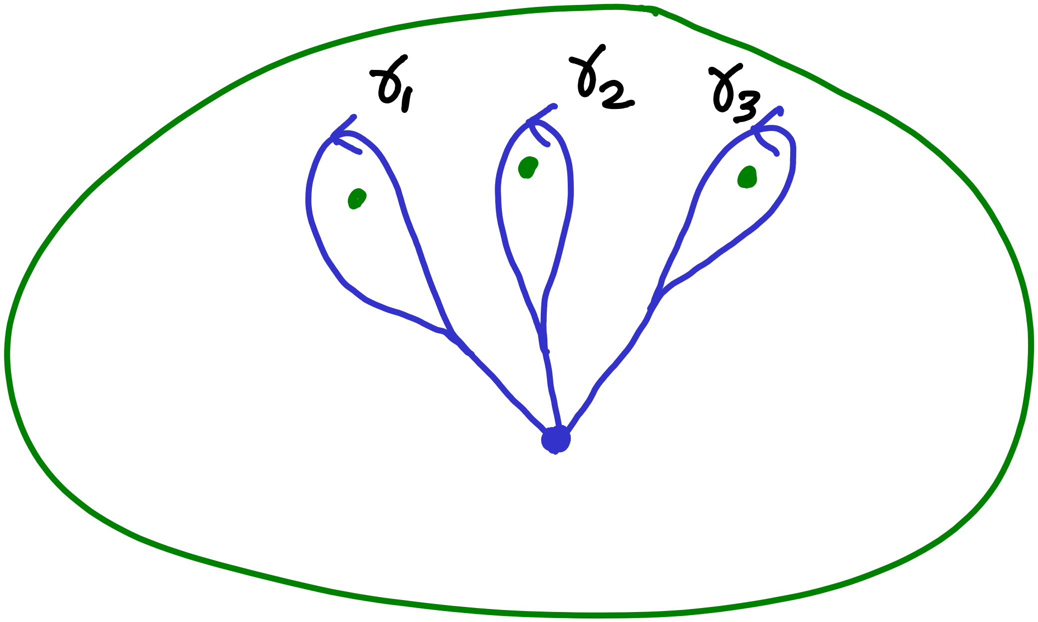
- second order linear algebraic differential equation
- singular points $0, 1, \infty \in \mathbb{P}^1(\mathbb{C})$

Riemann: Have basis of solutions on any disk $U \subset X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$

- Look at "monodromy" of bases of solutions around loops

$$\Rightarrow \rho \in \mathrm{Hom}(\pi_1(X, x), GL_2(\mathbb{C}))$$

$X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$



$M_i = \rho(\gamma_i)$

$$\mathrm{Hom}(\pi_1(X), GL_2(\mathbb{C})) \cong \{ M_1, M_2, M_3 \in GL_2(\mathbb{C}) \mid M_1 M_2 M_3 = 1 \}$$

- constants a, b, c ~ conjugacy classes of M_1, M_2, M_3
- conjugacy class of ρ in $\mathrm{Hom}(\pi_1, G)/G$ is intrinsic

(indept. of basepoint and initial basis)

More generally taking monodromy gives map:

Order n linear differential equations with singular points $a_1, \dots, a_m$

$$\Big\Downarrow \qquad \text{"Riemann–Hilbert map"}$$

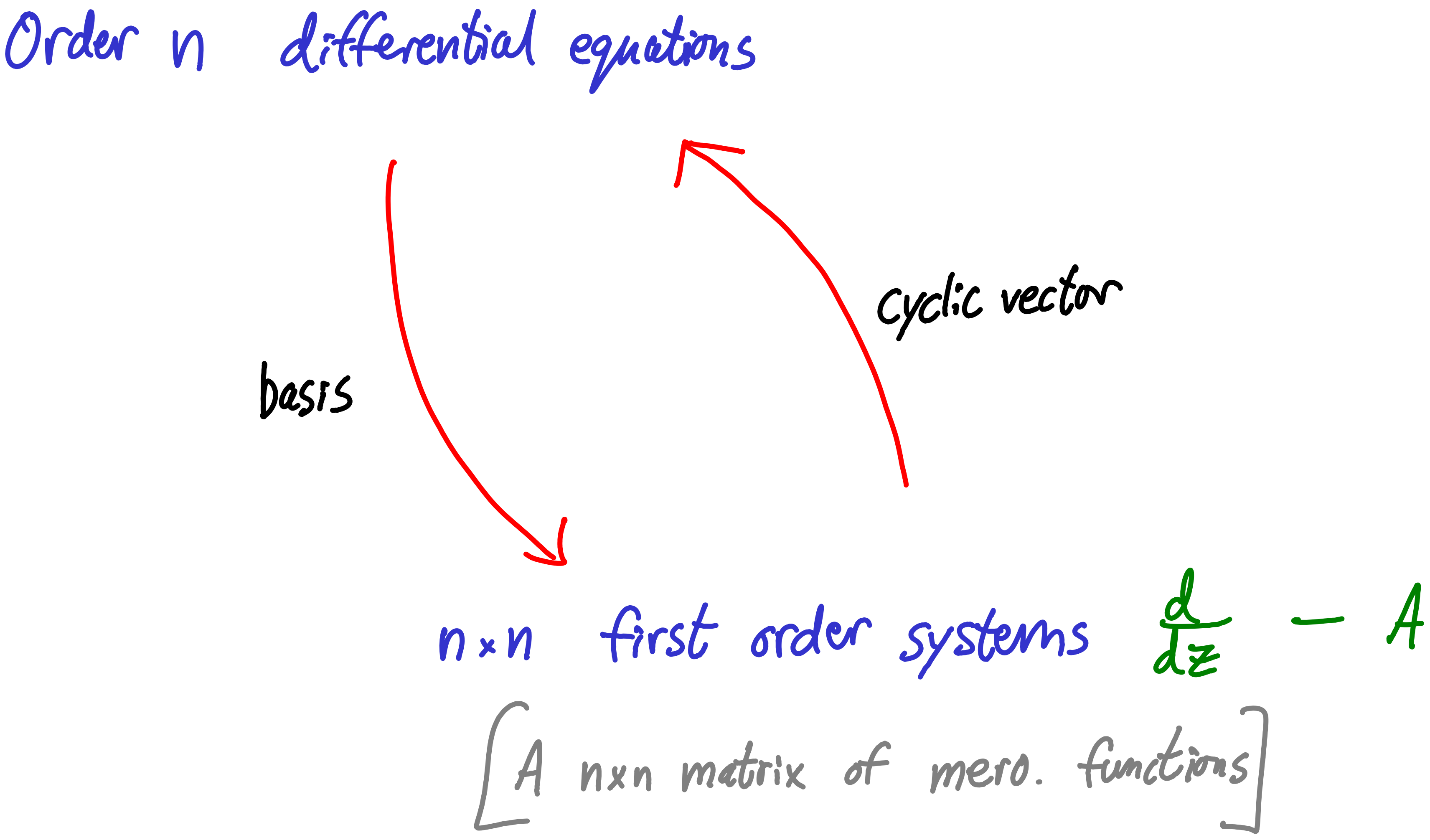
Point of $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}), GL_n(\mathbb{C})) / GL_n(\mathbb{C})$

Hilbert's 21st problem (modern restatement):

What's going on here?

– is there a precise correspondence here somewhere?

Evolution ①



Order n differential equations

basis

cyclic vector

n×n first order systems $\frac{d}{dz} - A$

[A n×n matrix of mero. functions]

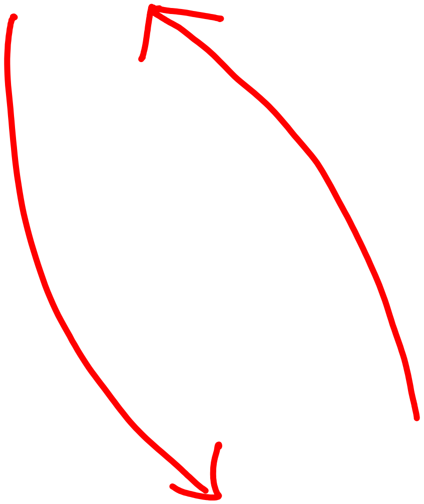
Evolution ②

n × n first order systems $\frac{d}{dz} - A$

connections on trivial rank n vector bundle (on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$) $d - A\,dz$

Evolution ②

$n \times n$ first order systems $\frac{d}{dz} - A$



connections on trivial rank n vector bundle

(on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

$$\begin{gathered} d - A\,dz \\ \parallel \\ d - B \end{gathered}$$

[B $n \times n$ matrix of mero. one-forms]

Evolution ②

n × n first order systems $\frac{d}{dz} - A$

connections on trivial rank n vector bundle (on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

$$\begin{array}{c} d - A\,dz \\ \| \\ d - B \end{array}$$

[B n×n matrix of mero. one-forms]

Locally have fundamental solutions $\Phi : U \to GL_n(\mathbb{C})$

$$\boxed{d\Phi = B\,\Phi}$$

Example $a_1, \dots, a_m \in \mathbb{C}$ $A_1, \dots, A_m \in \mathrm{End}(\mathbb{C}^n)$

$$d - \sum_1^m \frac{A_i}{z-a_i}\, dz \qquad \sum A_i = 0 \quad (\text{no pole at } \infty)$$

$$\Big\Downarrow \quad \mathrm{RH}$$

$$\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}), GL_n(\mathbb{C}))$$

$$\{ M_1, \dots, M_m \in GL_n(\mathbb{C}) \mid M_1 \cdots M_m = 1 \}$$

Theorem (Bolibruch)

This Riemann-Hilbert map is not surjective in general

Evolution ③

connections on trivial rank n vector bundle
(on $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$)

$$\nabla = d - B$$

connections ∇ on rank n vector bundles V
(on $\Sigma \setminus \{a_1, \dots, a_m\}$)
Σ genus g Riemann surface

$$\nabla : V \longrightarrow V \otimes \Omega^1$$

$$\nabla(fs) = (df)s + f(\nabla s)$$

Locally: $\nabla = d - B$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured Riemann surface

$G = GL_n(\mathbb{C})$

$$\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ holom. vector bundle} \\ \nabla \text{ holom. connection} \end{array} \right\} \Big/ \text{isom.} \quad \xleftrightarrow[\cong]{\substack{\text{analytic} \\ \text{Riemann-Hilbert}}} \quad \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

$$\begin{tikzcd}
\left\{ (\overline{V},\overline{\nabla}) \;\middle|\; \begin{array}{l} \overline{V}\to\overline{\Sigma} \text{ alg. vector bundle} \\ \overline{\nabla} \text{ mero. connection} \\ \quad \text{with } \underline{\text{simple poles}} \text{ at } \{a_i\} \end{array} \right\} \Big/ \text{isom.} \arrow[d, "\text{restrict to } \Sigma"'] & \\
\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V\to\Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \quad \text{with } \underline{\text{regular}} \text{ sing.s} \end{array} \right\} \Big/ \text{isom.} \arrow[r, leftrightarrow, "\overset{\text{Deligne Riemann-Hilbert}}{\simeq}"] & \mathrm{Hom}(\pi_1(\Sigma), G)/G
\end{tikzcd}$$

- Representations of π_1 classify algebraic differential equations (in this sense)
- similarly for any smooth quasi-proj. var. (Deligne) < add "flat/integrable"; simple poles ⟶ Logarithmic
- can now study transcendental aspects of RH map

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

$$\begin{tikzcd}
\left\{ (\overline{V},\overline{\nabla}) \,\middle|\, \begin{array}{l} \overline{V}\to\overline{\Sigma} \text{ alg. vector bundle} \\ \overline{\nabla} \text{ mero. connection} \\ \text{with } \underline{\text{poles}} \text{ at } \{a_i\} \end{array} \right\} \Big/ \text{isom.} \arrow[d, "\text{restrict to } \Sigma"'] & \\
\left\{ (V,\nabla) \,\middle|\, \begin{array}{l} V\to\Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \end{array}\right\} \Big/ \text{isom.} \arrow[r, leftrightarrow, "\text{Irregular Riemann-Hilbert}", "\simeq"'] & \{ \quad ? \quad \}
\end{tikzcd}$$

Aside: Applications / link to modern moduli theory

E.g. $m = 0$ (no poles), Σ compact smooth complex algebraic curve

$$G = GL_n(\mathbb{C})$$

$$\begin{tikzcd}
\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \end{array} \right\} \Big/ \text{isom.} \arrow[rr, leftrightarrow, "RH"', "\cong"] \arrow[d, "\pi"', "\text{forget } \nabla"] & & \mathrm{Hom}(\pi_1(\Sigma), G) / G \\
\left\{ \text{Alg. vector bundles } V \to \Sigma \right\} \big/ \text{isom.} & & \mathrm{Hom}(\pi_1(\Sigma), U_n) / U_n \arrow[ll] \arrow[u, hook]
\end{tikzcd}$$

A. Weil: ① π is not onto

② $\pi \circ RH$ is injective on unitary representations

"Weil's unitary trick"

Aside: Applications / link to modern moduli theory

E.g. $m=0$ (no poles), Σ compact smooth complex algebraic curve

$$G = GL_n(\mathbb{C})$$

$$\begin{tikzcd}
\left\{(V,\nabla) \middle| \begin{array}{l} V\to\Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection}\end{array}\right\}/\text{isom.} \arrow[rr, leftrightarrow, "\overset{RH}{\cong}"] \arrow[d, "\pi"', "\text{forget } \nabla"] & & \mathrm{Hom}(\pi_1(\Sigma), G)/G \\
\{\text{Alg. vector bundles } V\to\Sigma\}/\text{isom.} & & \mathrm{Hom}(\pi_1(\Sigma), U_n)/U_n \arrow[ll, hook'] \arrow[u, hook] \\
\cup & & \cup \\
\left\{\begin{array}{l}\text{Stable} \quad (\text{rk } n,\ \deg 0) \\ \text{alg. vector bundles } V\to\Sigma\end{array}\right\}/\text{isom.} \arrow[rr, leftrightarrow, "\cong"] & & \mathrm{Hom}^{irr}(\pi_1(\Sigma), U_n)/U_n
\end{tikzcd}$$

Mumford

Narasimhan–Seshadri

$V \to \Sigma$ is stable if

$$\frac{\deg(W)}{\operatorname{rank}(W)} < \frac{\deg(V)}{\operatorname{rank}(V)}$$

for any sub-bundle W of V

Thm (Mumford) $\{$stable vector bundles of rank n deg $d\}/_{\text{isom.}}$ forms a smooth quasi-projective algebraic variety

- so (via Naras.-Sesh.) $\operatorname{Hom}^{irr}(\pi_1(\Sigma), U_n)/U_n$ is Kähler

(complex manifold + compatible symplectic structure)

- similarly $\operatorname{Hom}^{irr}(\pi_1(\Sigma), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ is hyperKähler

(Hitchin, Donaldson, Corlette, Simpson)

$\mathrm{Hom}^{irr}(\pi_1(\Sigma), GL_n(\mathbb{C}))/GL_n(\mathbb{C})$ is hyperkähler

(Hitchin, Donaldson, Corlette, Simpson)

So has family of complex structures (& compatible symplectic structures)

Here only two complex structures are not isomorphic:

① as complex algebraic connections or complex π_1 representations

② as a moduli space of stable Higgs bundles $\sim T^*\{\text{stable vector bundles}\}$

$$(E, \Phi) \begin{cases} E \to \Sigma & \text{holom. vector bundle} \\ \Phi : E \to E \otimes \Omega^1 & (\mathcal{O}\text{-linear}) \end{cases}$$

$$\Phi(fs) = f\,\Phi(s) \qquad \text{(degen. Leibniz)}$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\begin{tikzcd}
\Sigma \arrow[r, Rightarrow] & H^1(\Sigma, G) \arrow[r, dash] \arrow[dr, dash] & \{\text{connections}\,(V,\nabla)\}/\text{isom.} \arrow[d, equal, "\text{RH correspondence}"] \\
\text{smooth compact curve} & & \mathrm{Hom}(\pi_1(\Sigma), G)/G
\end{tikzcd}$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

Σ $\Longrightarrow$

smooth compact curve

$$\mathcal{M}_{DR}(\Sigma) = \left\{ \overset{\text{stable}}{\text{connections}} (V, \nabla) \right\} \Big/ \text{isom.}$$

$$\Big|\Big|\Big| \text{ RH isomorphism}$$

$$\mathcal{M}_B(\Sigma) = \mathrm{Hom}^{\mathrm{irr}}(\pi_1(\Sigma), G) \big/ G$$

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
 & \left\{\text{solutions to Hitchin's equations}\right\} \arrow[d, phantom, sloped, "="] & \\
\Sigma \arrow[r, Rightarrow] & \mathcal{M}(\Sigma) \arrow[r, phantom, "\cong", "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, phantom, sloped, "\cong", "\text{RH isomorphism}"'] \\
\text{smooth compact curve} & \text{Hyperkahler manifold} & \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma) = \left\{ \overset{\text{stable}}{\text{Higgs bundles}} \; (E,\Phi) \right\} \Big/ \text{isom.}$$

$$\begin{tikzcd}
\Sigma \arrow[r, Rightarrow] & \mathcal{M}(\Sigma) \arrow[r, phantom, "\cong", "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, phantom, "\cong"', "\text{RH isomorphism}"] \\
 & & \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

smooth compact curve

Hyperkahler manifold

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
 & & \mathcal{M}_{Dol}(\Sigma) \\
\Sigma \arrow[r, Rightarrow] & \mathcal{M}(\Sigma) \arrow[ur, equal, "\text{Hitchin-Simpson}"] \arrow[r, equal, "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, equal, "\text{RH isomorphism}"] \\
\text{smooth compact curve} & \text{Hyperkahler manifold} & \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

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Fix $G = GL_n(\mathbb{C})$

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 & & \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

smooth compact curve

Hyperkahler manifold

3 algebraic structures

so for $m=0$ get a rich picture:

"Non-abelian Hodge package"

Fix $G = GL_n(\mathbb{C})$

$$\begin{tikzcd}
 & & \mathcal{M}_{Dol}(\Sigma) \arrow[d, equal, "\text{Non-abelian Hodge}"] \\
\Sigma \arrow[r, Rightarrow] & \mathcal{M}(\Sigma) \arrow[ur, equal, "\text{Hitchin-Simpson}"] \arrow[r, equal, "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, equal, "\text{RH isomorphism}"] \\
 & & \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

smooth compact curve

Hyperkahler manifold

3 algebraic structures

- Similarly for π_1(punctured curve)

(Simpson, Konno, Nakajima ...)

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
& \mathcal{M}_{Dol}(\Sigma) \arrow[d, equal, "\text{Non-abelian Hodge}"] \\
\mathcal{M}(\Sigma) \arrow[ur, equal, "\text{Hitchin-Simpson}"] \arrow[r, equal, "\text{Corlette-Donaldson}"'] & \mathcal{M}_{DR}(\Sigma) \arrow[d, equal, "\text{RH isomorphism}"] \\
& \mathcal{M}_B(\Sigma)
\end{tikzcd}$$

yperkahler
manifold

3 algebraic structures

ured curve)

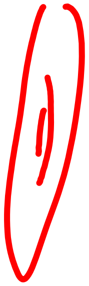
gma ...)

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
\mathcal{M}_{Dol}(\Sigma) \arrow[r, dash] \arrow[d, "\cong"', "\text{Non-abelian Hodge}"] & \text{Algebraic integrable Hamiltonian systems (Hitchin)} \\
\mathcal{M}_{DR}(\Sigma) \arrow[d, "\cong"', "\text{RH isomorphism}"] & \\
\mathcal{M}_{B}(\Sigma) &
\end{tikzcd}$$



$$\begin{tikzcd}
{} \arrow[r, hook] & \mathcal{M}_{Dol}(\Sigma) \arrow[d] \\
& \mathbb{H}
\end{tikzcd}$$

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
\mathcal{M}_{Dol}(\Sigma) \arrow[d, phantom, "\cong" description, "\text{Non-abelian Hodge}"'] \\
\mathcal{M}_{DR}(\Sigma) \arrow[r, dash] \arrow[d, phantom, "\cong" description, "\text{RH isomorphism}"'] & \text{Isomonodromy systems (as } \Sigma \text{ varies)} \\
\mathcal{M}_B(\Sigma)
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{\sim}{\Sigma} \arrow[d] & \Longrightarrow & \mathcal{M}_{DR}(\Sigma_b) \arrow[d] & \subset & \mathcal{M}_{DR/\mathbb{B}} \arrow[d] \arrow[r, dash] & \text{fibre bundle with flat nonlinear connection} \\
\mathbb{B} & & b & \in & \mathbb{B}
\end{tikzcd}$$

e.g. Painlevé VI equations, Schlesinger system

"nonabelian Gauss-Manin connection"

so for $m=0$ get a rich picture:

Fix $G = GL_n(\mathbb{C})$

"Non-abelian Hodge package"

$$\begin{tikzcd}
\mathcal{M}_{Dol}(\Sigma) \arrow[d, phantom, "\cong" description, "\text{Non-abelian Hodge}" right] \\
\mathcal{M}_{DR}(\Sigma) \arrow[d, phantom, "\cong" description, "\text{RH isomorphism}" right] \\
\mathcal{M}_B(\Sigma) \arrow[r, dash] & \text{Nonlinear braid/mapping class group actions}
\end{tikzcd}$$

$\pi_1(\mathcal{M}_{g,m})$

$$\begin{tikzcd}
\underset{\sim}{\Sigma} \arrow[d] \\
\mathbb{B}
\end{tikzcd} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowleft \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

End of Aside

Beyond the fundamental group

$\Sigma = \overline{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve$/\mathbb{C}$

$G = GL_n(\mathbb{C})$

$$\left\{ (V,\nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \end{array} \right\} \Big/ \text{isom.} \quad \overset{\text{Irregular Riemann-Hilbert}}{\overset{\cong}{\longleftrightarrow}} \quad \left\{ \quad ? \quad \right\}$$

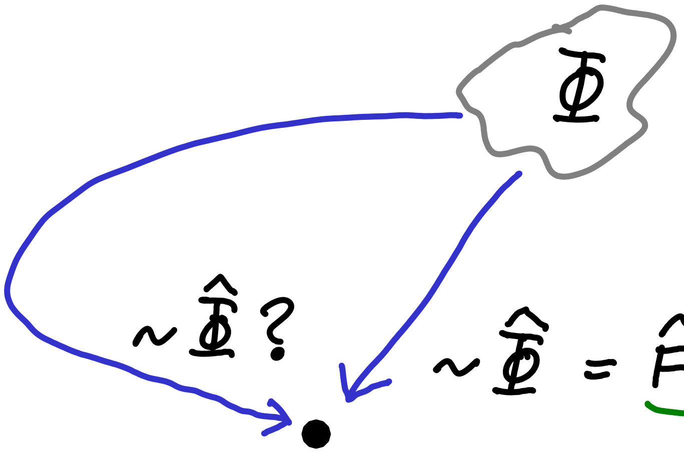
Want to generalise "monodromy representations"

to classify irregular connections/differential equations

typically poles of order ≥ 2, e.g. equations of Airy, Bessel, Whittaker, Kummer...

Stokes phenomenon

$$\nabla = d - \left(\frac{A_k}{z^k} + \dots\right) dz$$



$$\sim \hat{\Phi} = \underbrace{\hat{F}(z)}_{\text{divergent power series}} z^{\Lambda} \exp\left(\overbrace{Q(1/z)}^{Q = \sum_i^r B_i/z^i}\right)$$

Stokes phenomenon

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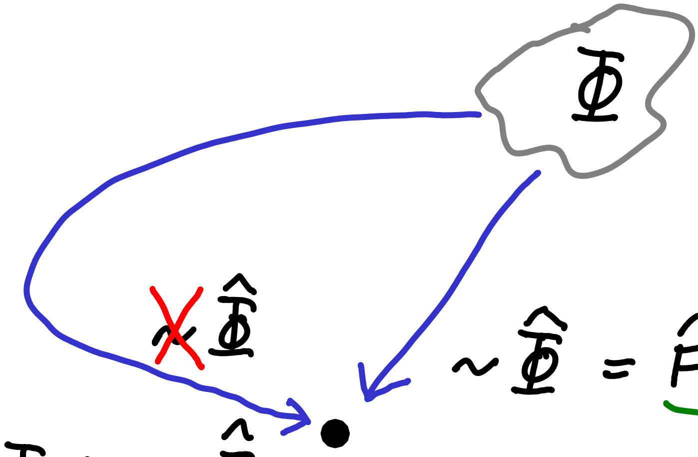

$$\sim \hat{\Phi} = \underbrace{\hat{F}(z)}_{\text{divergent power series}} z^{\Lambda} \exp\Big(\overbrace{Q(1/z)}^{Q = \sum_1^r B_i/z^i}\Big)$$

but $\exists$ different solution $\Phi \underbrace{C}_{\text{constant } \in Gl_n(\mathbb{C})} \sim \hat{\Phi}$

"Stokes matrix"

"Irregular connections can be classified by their Stokes & monodromy data"

Birkhoff, Jurkat, Sibuya, Malgrange, Ramis, Deligne, . . .

Main results

① Stokes & monodromy data form nice spaces generalising $\mathrm{Hom}(\pi_1(\Sigma), G)/G = \mathcal{M}_B(\Sigma)$

"Wild character varieties"

- holomorphic symplectic manifolds (irregular Atiyah-Bott)
- algebraic symplectic manifolds (extend q-Hamiltonian approach)
- complete hyperkahler manifolds (with O. Biquard)
  - including some new gravitational instantons
  - wild nonabelian Hodge correspondence (whole package)

② Q behaves like moduli of Riemann surface with marked points

$\hat{F}(z) z^{\Lambda} \exp(Q(1/z))$

- notion of "wild Riemann surface" Σ
- notion of "moduli space of wild Riemann surfaces" $\mathcal{M}(\Sigma)$
- notion of "wild mapping class group" $\Gamma(\Sigma) = \pi_1(\mathcal{M}(\Sigma))$

$\supset$ all G-braid groups (G simple)

Theorem Let Σ be a wild Riemann surface, with wild mapping class group $\Gamma(\Sigma)$ and wild character variety $\mathcal{M}_B(\Sigma)$

Then $\Gamma(\Sigma)$ acts on $\mathcal{M}_B(\Sigma)$ by algebraic Poisson automorphisms

Example

$$Q = A/z^2$$ (A diagonal with distinct eigenvalues)

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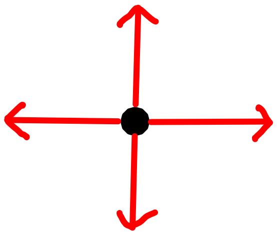
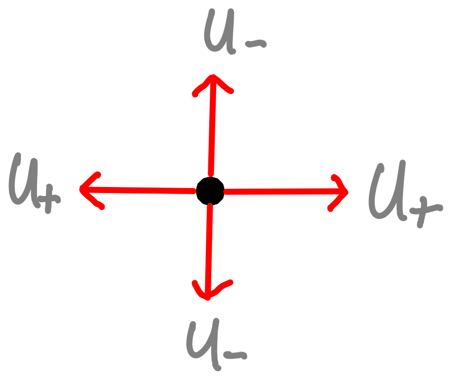


Singular directions

Example

$Q = A/z^2$ (A diagonal with distinct eigenvalues)



Singular directions

Stokes groups

Example

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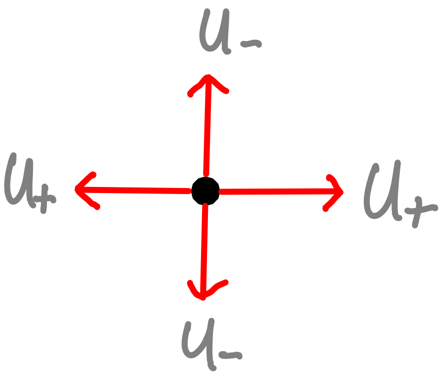
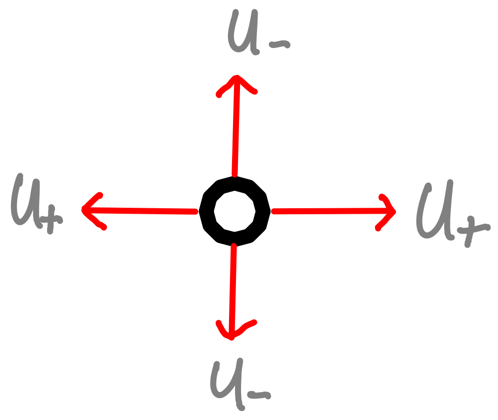


Singular directions

Stokes groups

$$U_+ = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

Example $Q = A/z^2$ (A diagonal with distinct eigenvalues)



Singular directions

Stokes groups

Real blow-up

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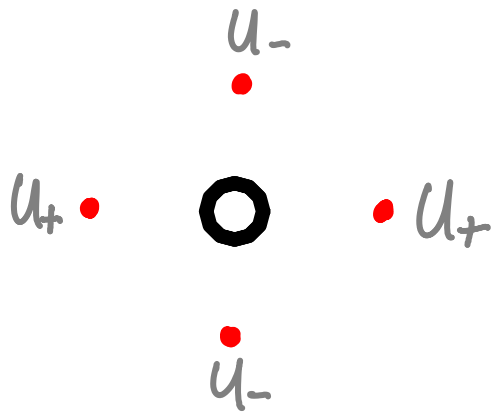

extra punctures

Stokes groups

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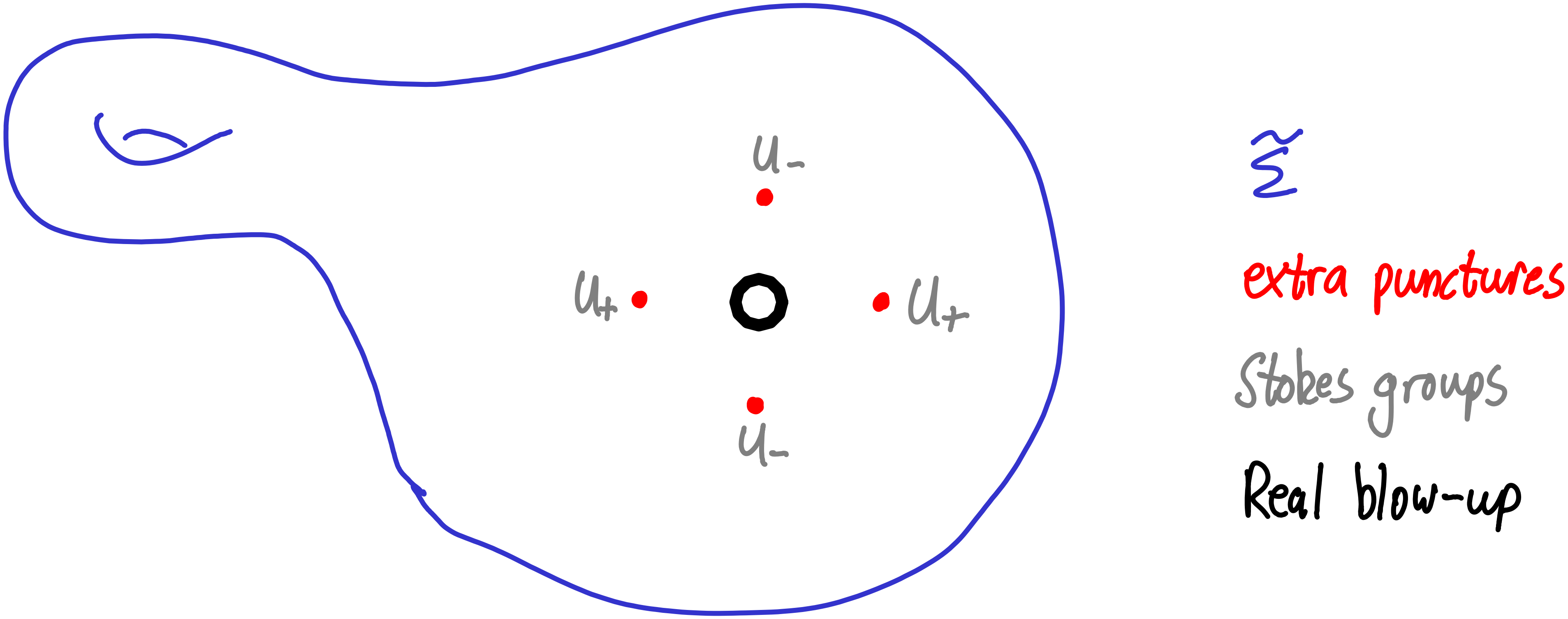
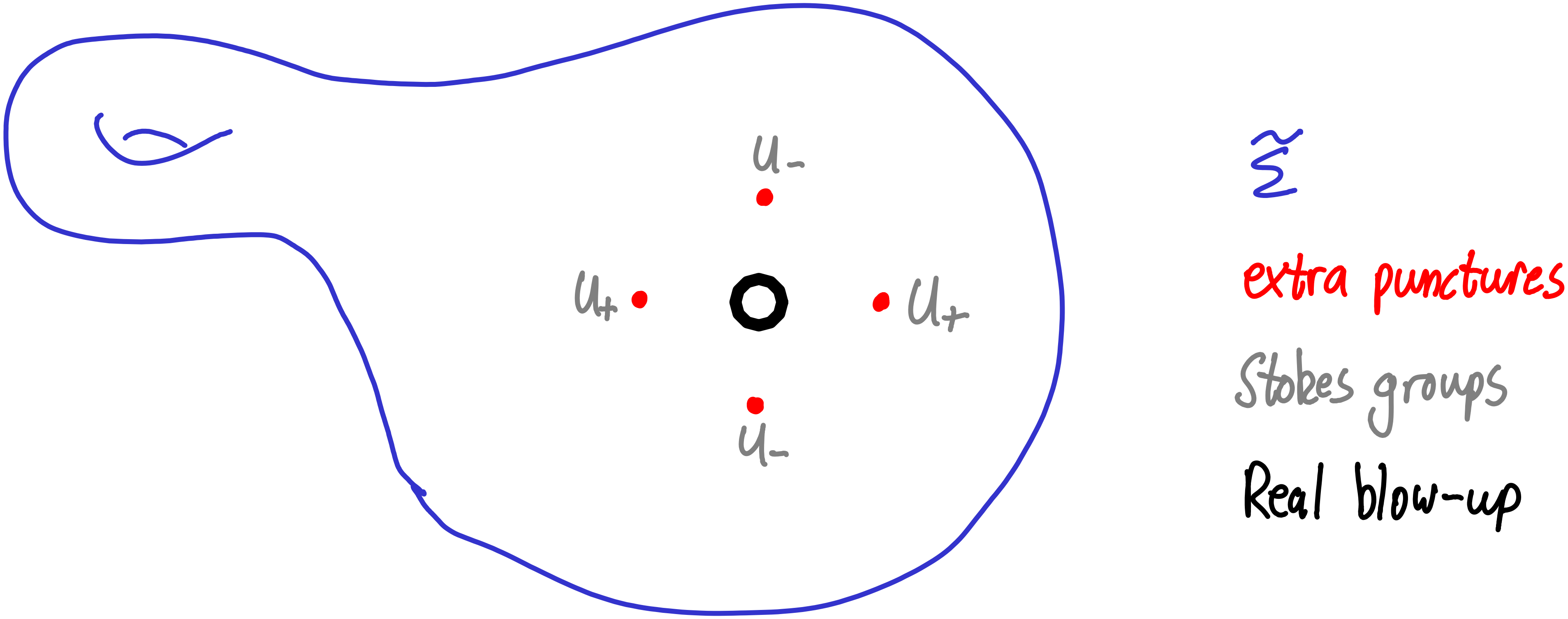


Example

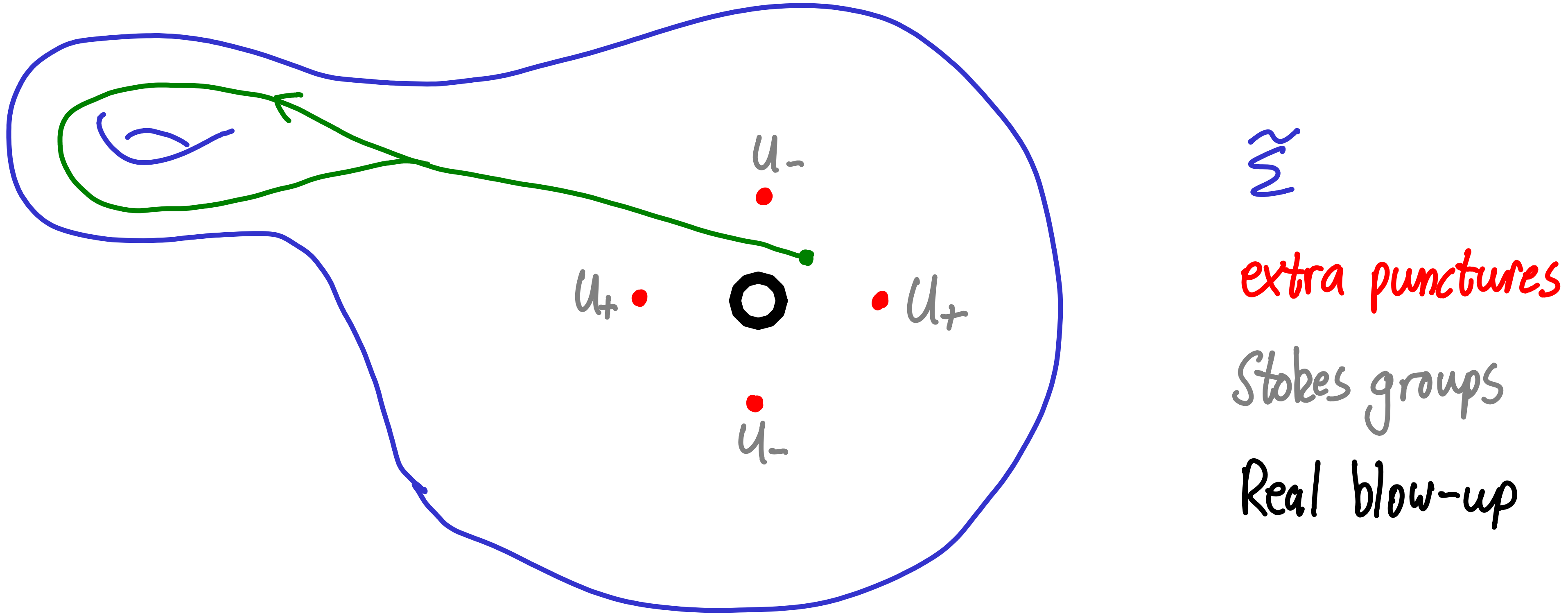
$$Q = A/z^2 \qquad (A \text{ diagonal with distinct eigenvalues})$$



Wild character variety $\mathcal{M}_B = \mathrm{Hom}_{\mathbb{S}}(\pi_1(\tilde{\Sigma}), GL_n(\mathbb{C}))/T$

Example

$Q = A/z^2$ (A diagonal with distinct eigenvalues)



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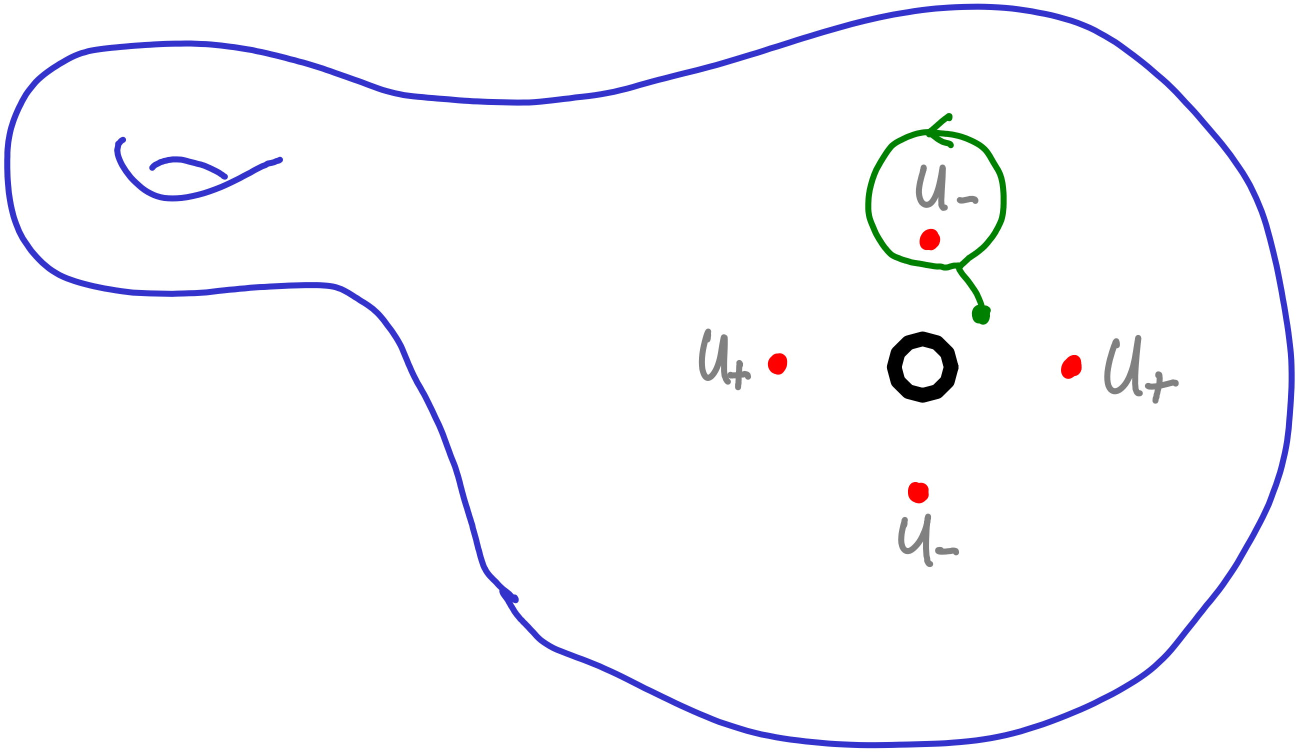

$\tilde{\Sigma}$

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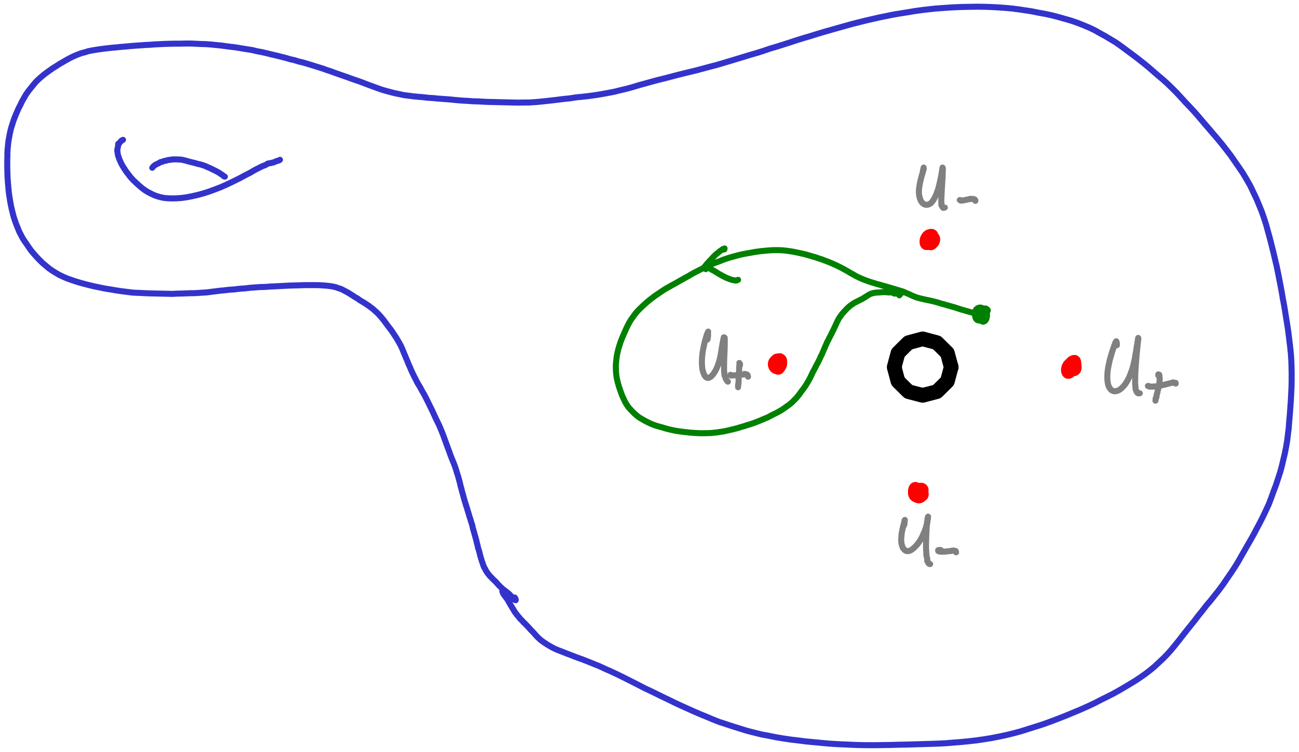

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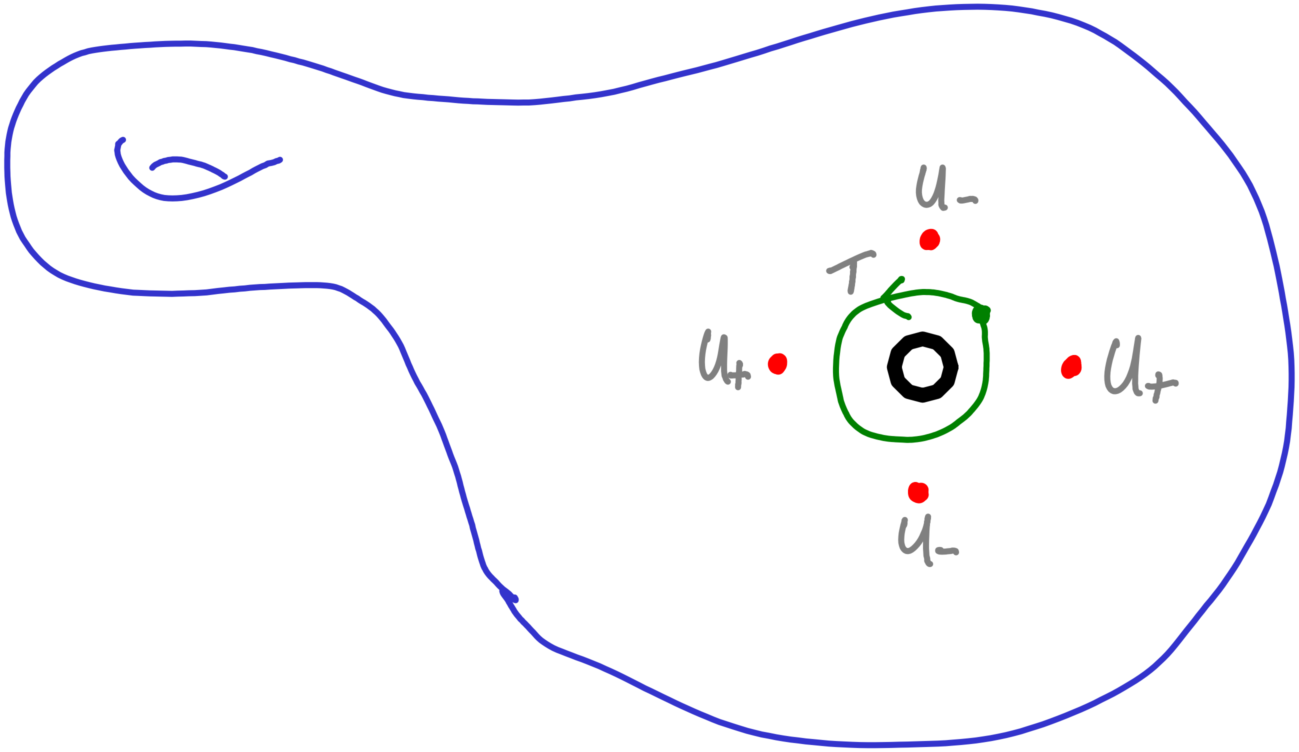

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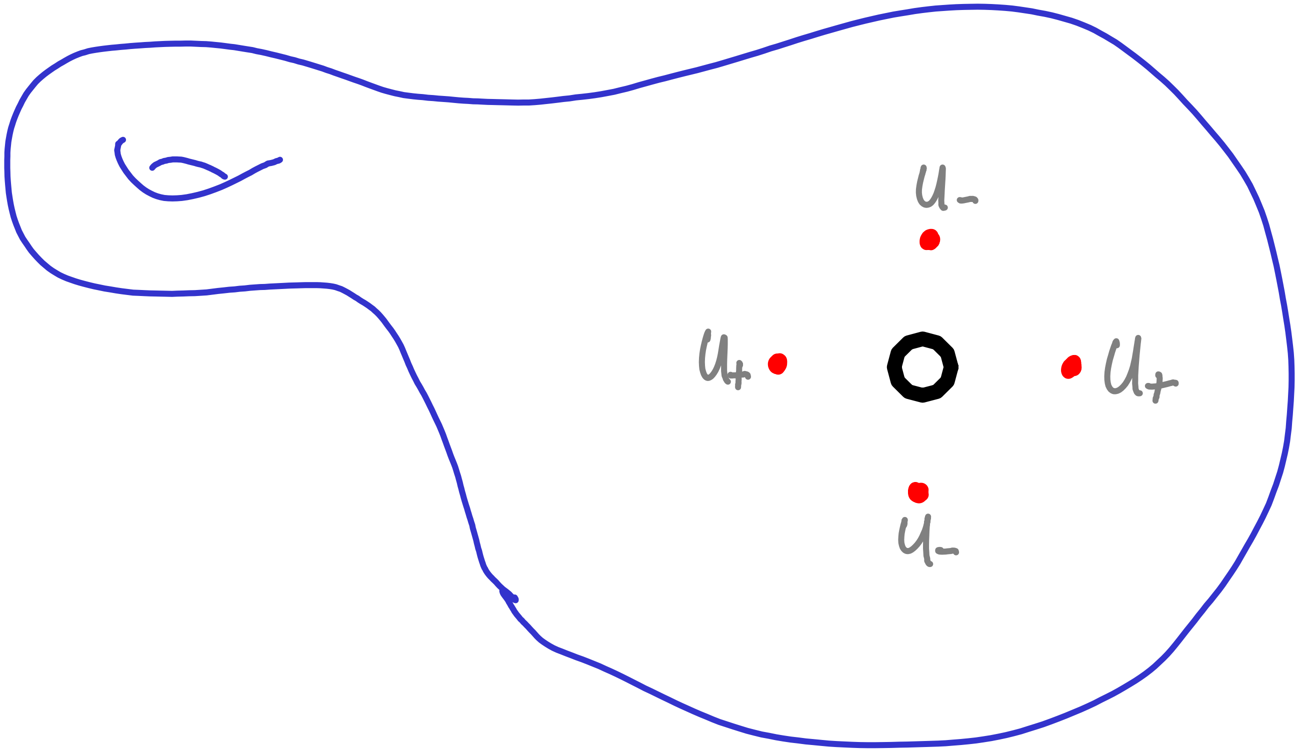
Wild character variety $\mathcal{M}_B = \mathrm{Hom}_{\mathbb{S}}(\pi_1(\tilde{\Sigma}), GL_n(\mathbb{C}))/T$

$$T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

Example

$Q = A/z^2$ (A diagonal with distinct eigenvalues)



$\tilde{\Sigma}$

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Stokes groups

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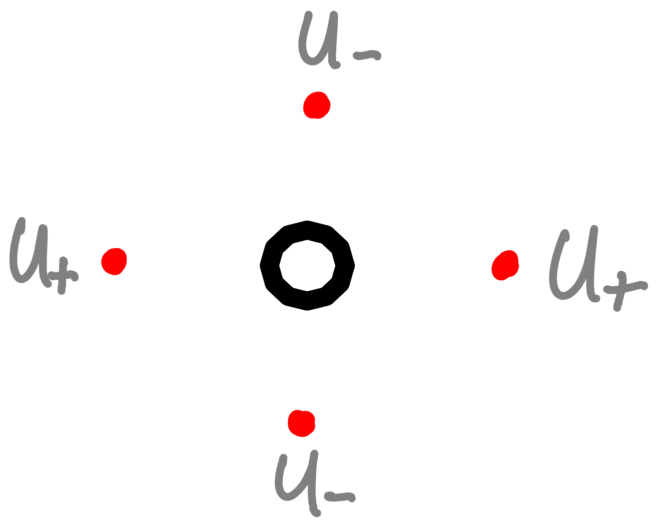

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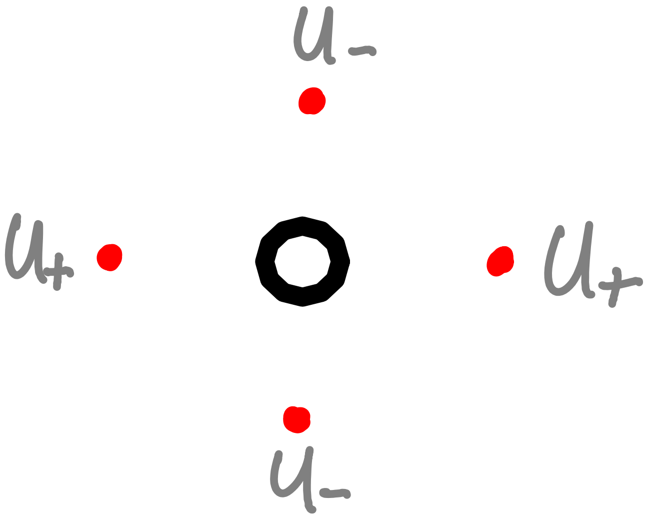
Example

$$Q = A/z^2$$ (A diagonal with distinct eigenvalues)

$$U_+ = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$



$\tilde{\Sigma}$

extra punctures

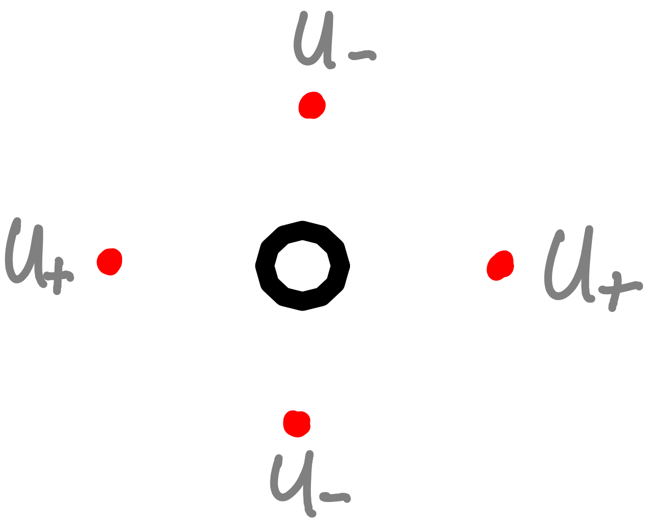
Stokes groups

Real blow-up

Example

$Q = A/z^2$ (A diagonal with distinct eigenvalues)

$$\mathcal{U}_+ = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$



$\tilde{\Sigma}$

extra punctures

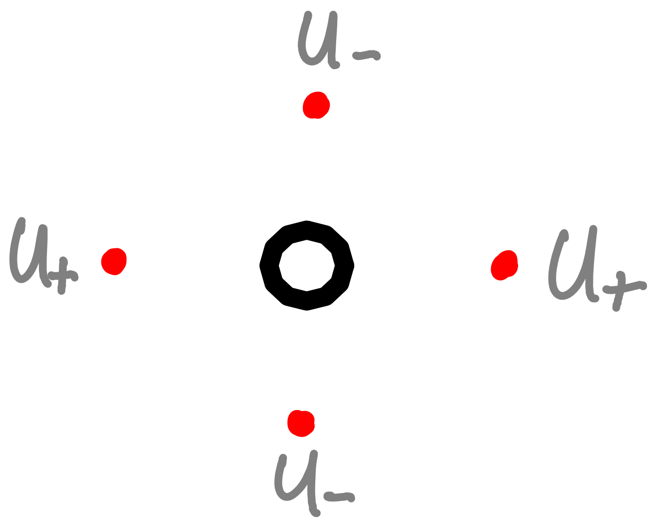
Stokes groups

Real blow-up

Example

$$Q = A/z^2 \qquad \text{(A diagonal with distinct eigenvalues)}$$

$$U_+ = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} = \quad U_{12} \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$



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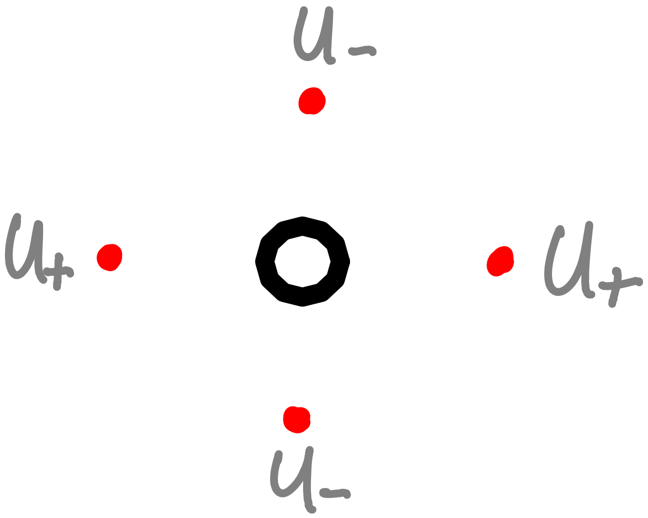

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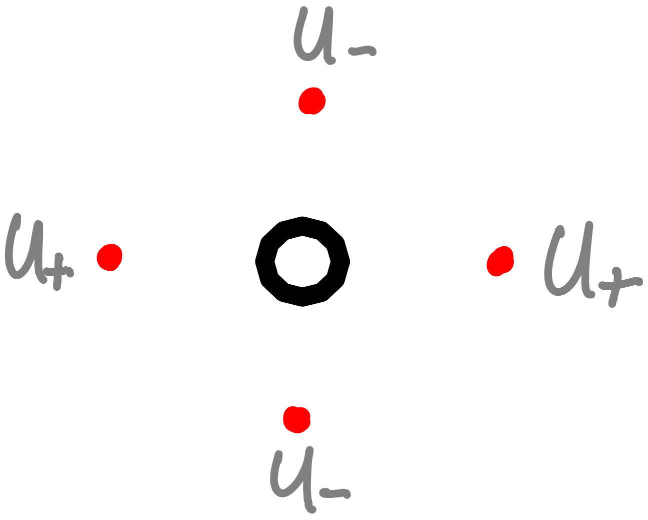
$Q = A/z^2$ (A diagonal with distinct eigenvalues)

$U_+ = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} = U_{12} \; U_{13} \; U_{23}$

- any order
- "direct spanning" subgroups



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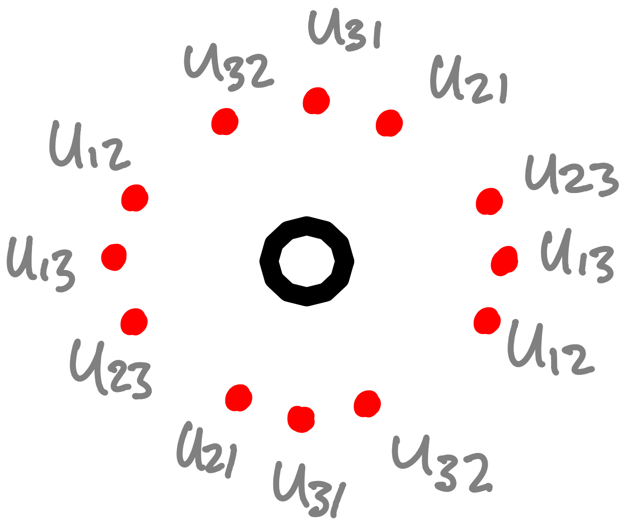

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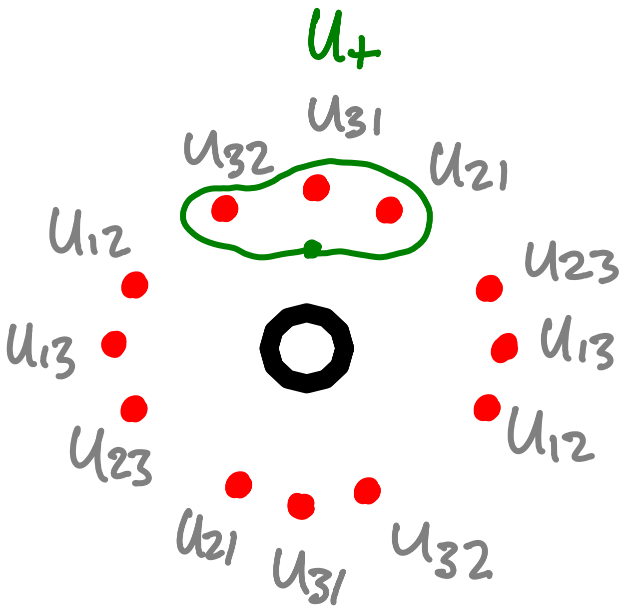

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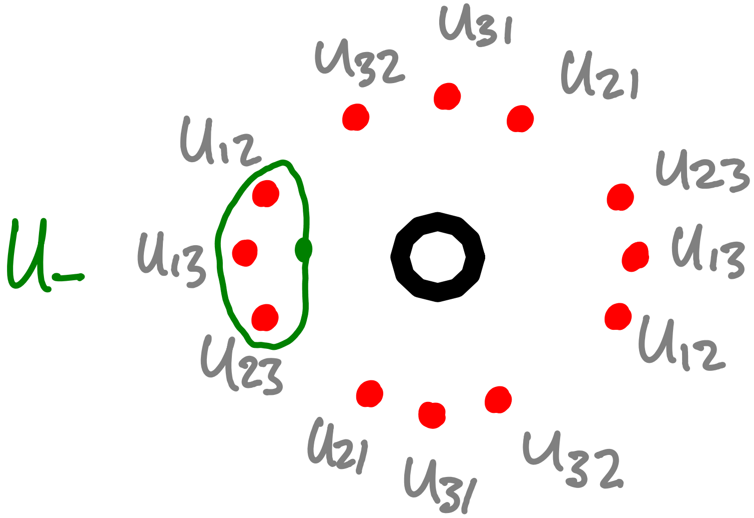

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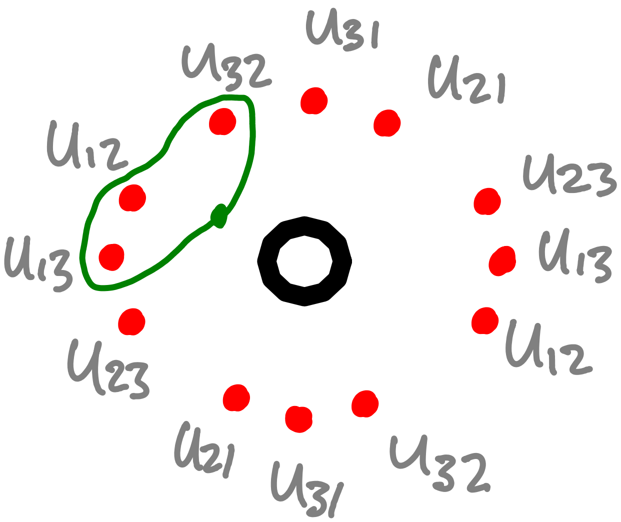

$\widetilde{\Sigma}$

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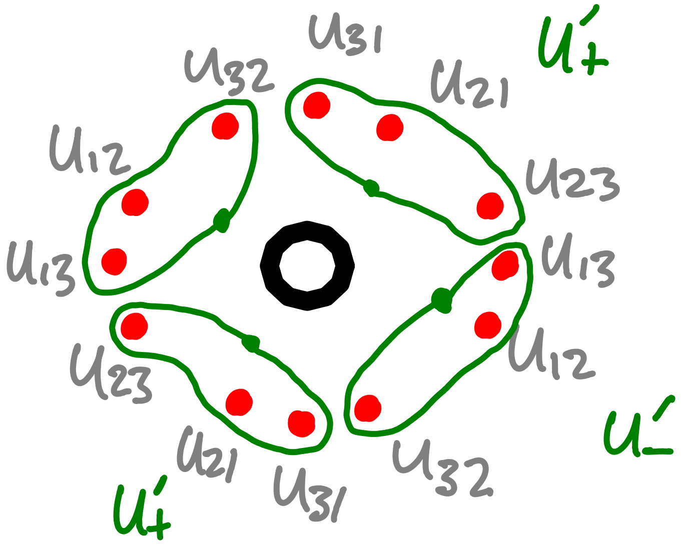

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Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

GLn other G

$$Q = A/z^2$$

(A diagonal with distinct eigenvalues)

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

GL_n other G

$$Q = -A/z$$

(A diagonal with distinct eigenvalues)

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
GL_n other G

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{\mathrm{reg}}\right)$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

GLn — other G

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$G = K_{\mathbb{C}}$ complex reductive group (e.g. $GL_n(\mathbb{C})$)

$T \subset G$ maximal torus

$$\mathfrak{g} = \text{Lie}(G) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

root spaces $\cong \mathbb{C}$

$\mathfrak{t} = \text{Lie}(T)$

roots $R \subset \mathfrak{t}^*$

$$\mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} \mid [X, Y] = \alpha(X) Y \quad \forall X \in \mathfrak{t} \}$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

GL_n / other G

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

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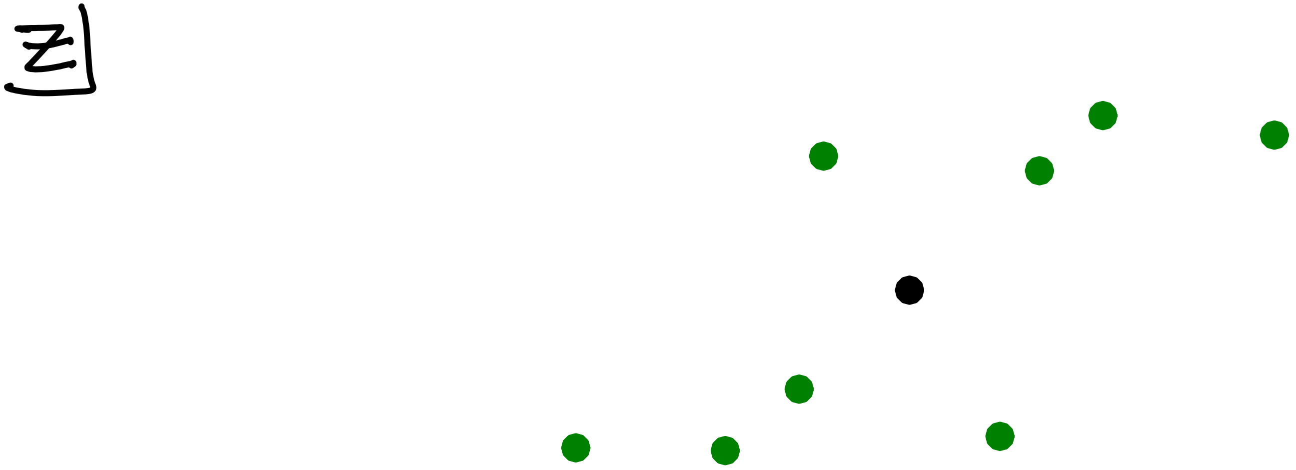

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

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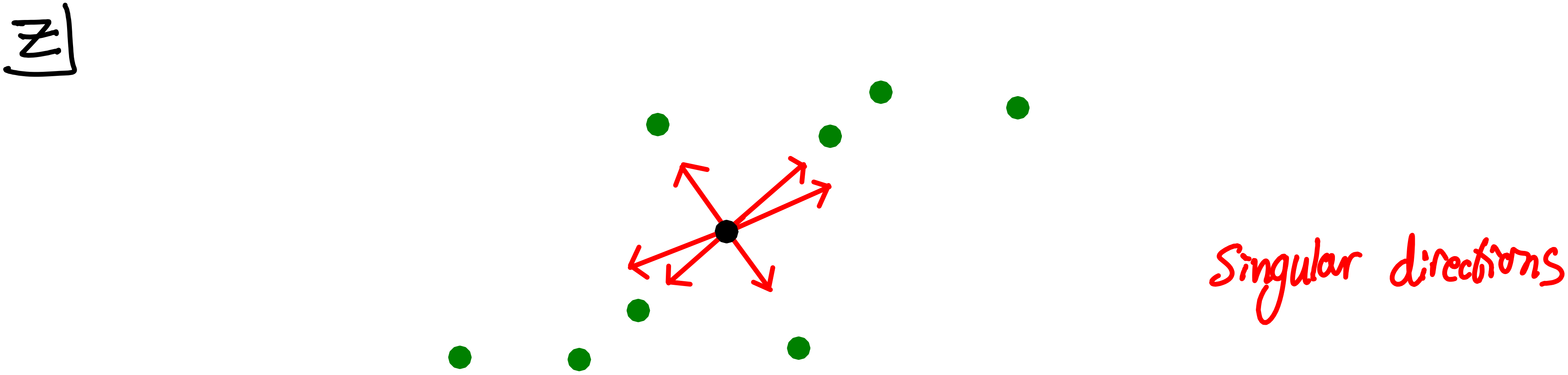

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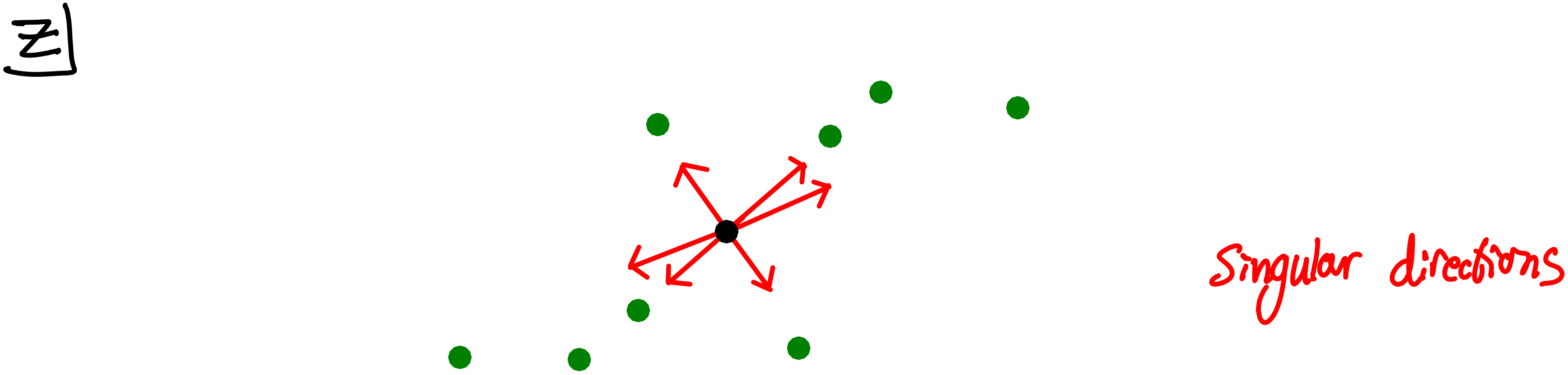

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser–Jurkat–Lutz '79, P.B. '01)

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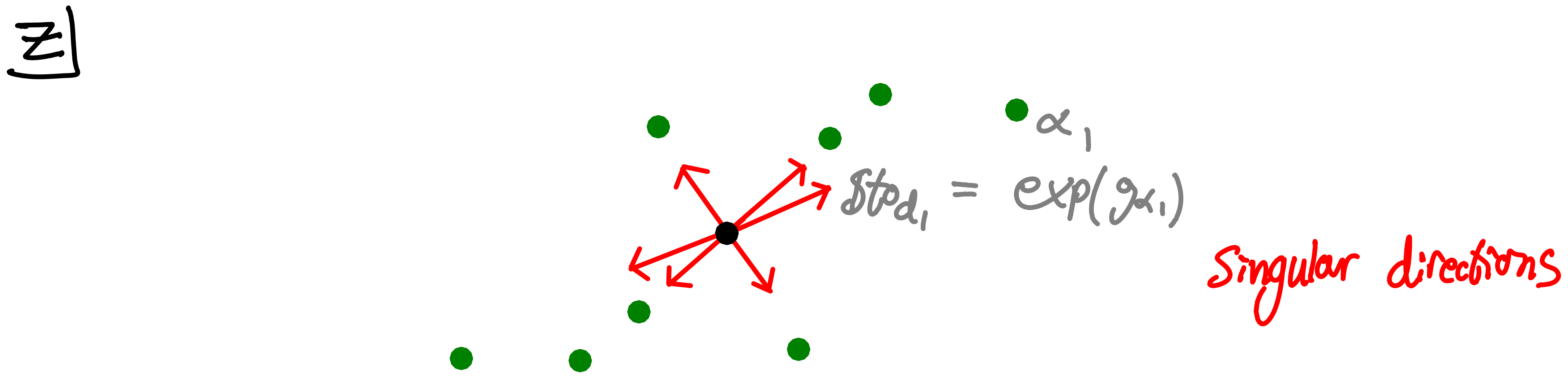

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

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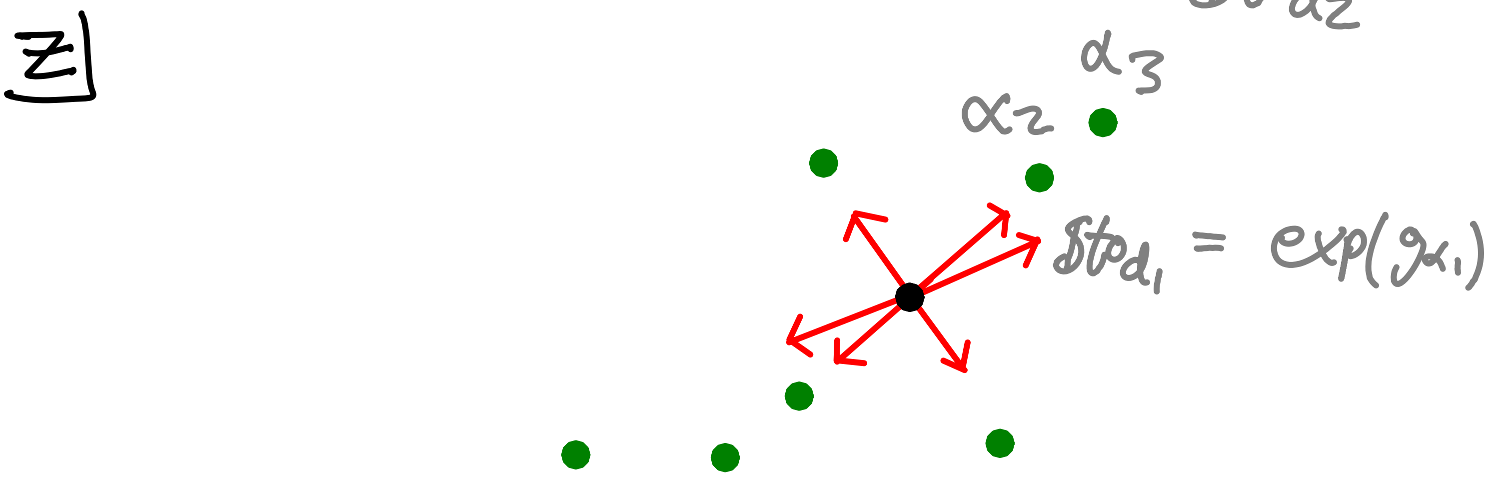
Example of definition of Stokes groups (Balser–Jurkat–Lutz '79, P.B. '01)

GL_n other G

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \,, \qquad R \subset \mathfrak{t}^*$$

$$\mathrm{Sto}_{d_2} = \exp(\mathfrak{g}_{\alpha_2})\exp(\mathfrak{g}_{\alpha_3})$$



$$\mathrm{Sto}_{d_1} = \exp(\mathfrak{g}_{\alpha_1})$$

Singular directions

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79 GL_n, P.B. '01 other G)

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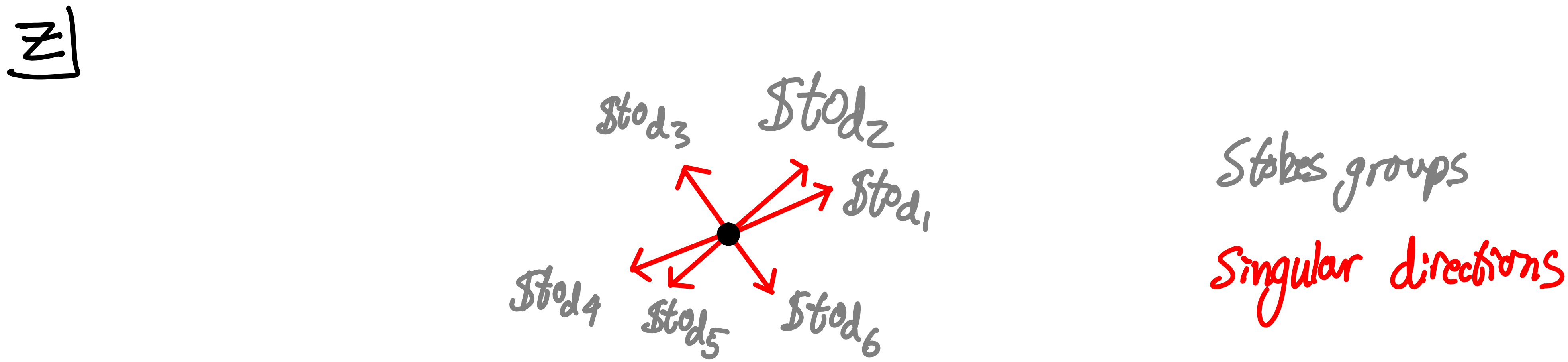

Plot $\langle R, A\rangle \subset \mathbb{C}^*$

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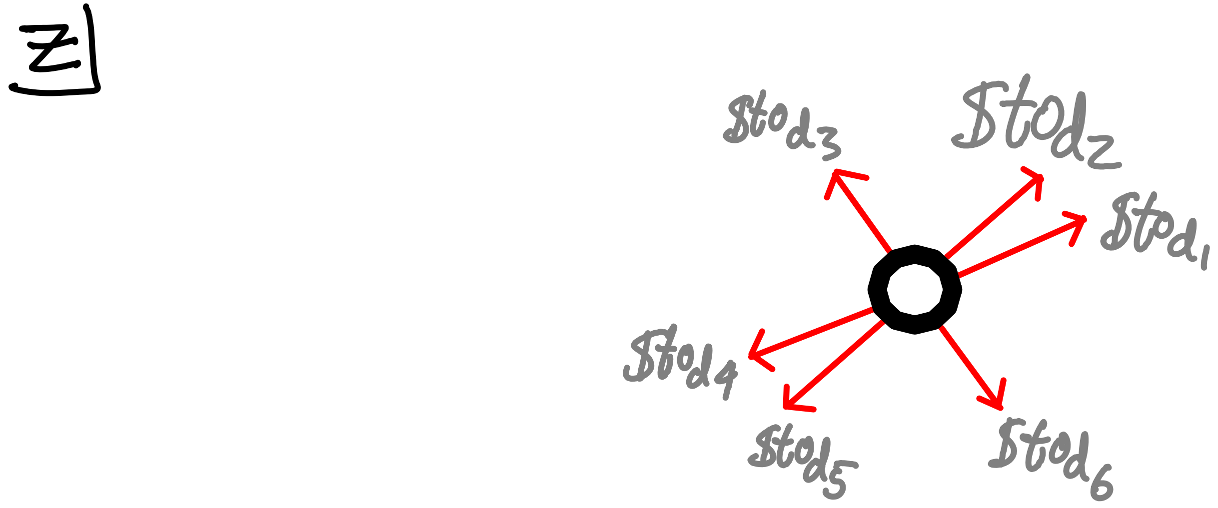

Stokes groups

Singular directions

Real blow-up

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser–Jurkat–Lutz '79, P.B. '01)

GLn, other G

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

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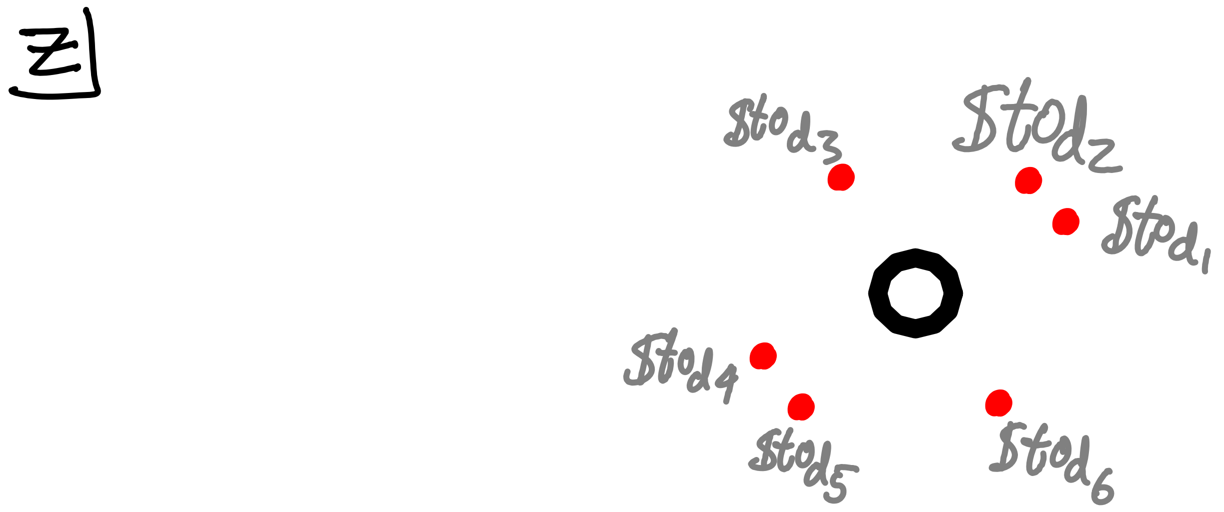


$\tilde{\Sigma}$

Stokes groups

extra punctures

Real blow-up

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

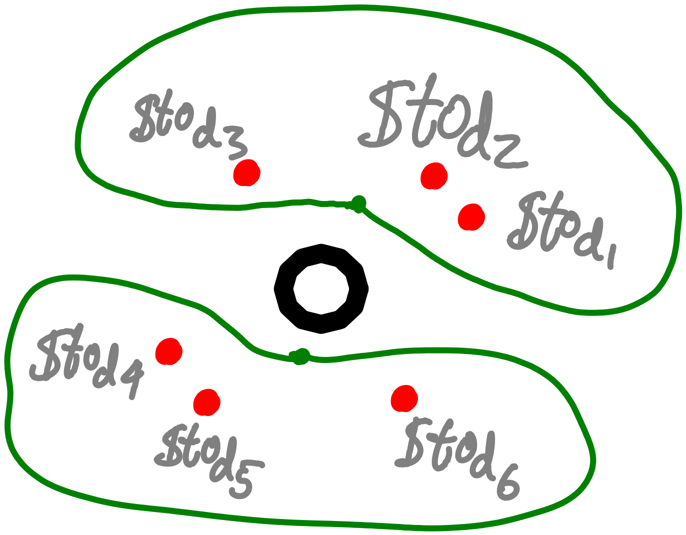
Example of definition of Stokes groups (Balser-Jurkat-Lutz '79 GL_n, P.B. '01 other G)

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad , \quad R \subset \mathfrak{t}^*$$

z

Half-periods ⇒ unipotent radicals of Borels



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extra punctures

Real blow-up

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$Sto_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

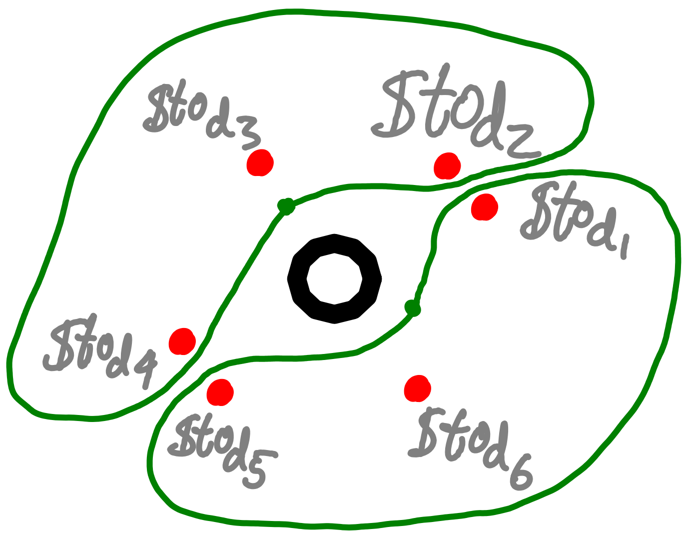
GL_n other G

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \,, \qquad R \subset \mathfrak{t}^*$$

$z\rfloor$

Half-periods ⇒ unipotent radicals of Borels



$\tilde{\Sigma}$

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$$\text{Plot } \langle R, A \rangle \subset \mathbb{C}^*$$

$$\mathbb{S}to_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79 GL_n, P.B. '01 other G)

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

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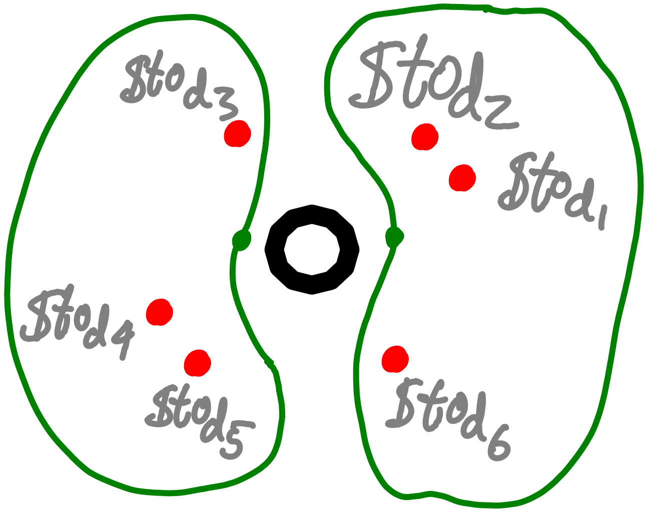
$\boxed{z}$

Half-periods ⇒ unipotent radicals of Borels



$\tilde{\Sigma}$

Stokes groups

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Real blow-up

Plot $\langle R, A\rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \mid \alpha(A)\in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$