Themes:

- (wild) non-abelian Hodge correspondence on curves & hyperkähler moduli spaces
- example moduli spaces on $\mathbb{P}^1$ $\quad (\mathcal{M}^*_{DR} \subset \mathcal{M}_{DR})$
- symplectic geometry of wild character varieties
- nonlinear symplectic braid group actions
- (Logahoric connections & Grothendieck-Brieskorn-Springer)

Wild nonabelian Hodge theory on curves

Choose
- $G = GL_n(\mathbb{C})$, $T \subset G$

"irregular curve" or "wild Riemann surface"
- Σ compact smooth complex algebraic curve
- $a_1, \dots, a_m \in \Sigma$ distinct points
- irregular types Q_i at a_i , $i = 1, \dots, m$

Definition If $a \in \Sigma$, an irregular type Q at a is an element $Q \in \mathfrak{t}(\hat{K}) / \mathfrak{t}(\hat{\mathcal{O}})$

If z is a local coordinate vanishing at a

$$\hat{\mathcal{O}} = \mathbb{C}[\![z]\!] , \quad \hat{K} = \mathbb{C}(\!(z)\!)$$

$$Q = \frac{A_r}{z^r} + \dots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t} = \mathrm{Lie}(T)$$

Wild nonabelian Hodge theory on curves

Choose

- $G = GL_n(\mathbb{C})$, $T \subset G$
- $\Sigma = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ irregular curve

Wild nonabelian Hodge theory on curves

Choose

- $G = GL_n(\mathbb{C})$, $T \subset G$
- $\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ irregular curve
- weights $\theta_1, \dots, \theta_m \in \mathfrak{t}_{\mathbb{R}} = X_*(T) \otimes \mathbb{R} \subset \mathfrak{t}$

$$\left((\theta_i)_{jj} \in [0,1) \quad j = 1, \dots, n \right)$$

Let $\mathfrak{h}_i = C_{\mathfrak{g}}(Q_i) \subset \mathfrak{g}$ (centraliser)

- adjoint orbits $\mathcal{O}_i \subset \mathfrak{l}_i := C_{\mathfrak{h}_i}(\theta_i) = C_{\mathfrak{g}}(Q_i, \theta_i)$

Note that $\theta \in \mathfrak{t}_{\mathbb{R}}$ determines a parabolic $\mathfrak{p}_\theta \subset \mathfrak{g}$

$$\mathfrak{p}_\theta(\mathfrak{g}) = \left\{ X \in \mathfrak{g} \;\middle|\; \lim_{z \to 0} z^{\theta} X z^{-\theta} \text{ along any ray exists} \right\}$$

Wild nonabelian Hodge theory on curves

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$$\mathfrak{p}_\theta(\mathfrak{g}) = \mathrm{Stab}(\mathcal{F}_\theta), \quad (\mathcal{F}_\theta)_\alpha = \bigoplus_{\beta \geq \alpha} E_\beta \quad \begin{pmatrix}\text{eigenspaces} \\ \text{of } \theta\end{pmatrix}$$

& similarly $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \subset \mathfrak{h}_i$ & $\mathfrak{l}_i$ is Levi of $\mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$

Consider triples $(V, \nabla, \mathcal{F})$

- $V \to \Sigma$ rank n holom. vector bundle
- $\nabla : V \to V \otimes \Omega^1(*D)$ mero. connection $D = \sum a_i$
- $\mathcal{F} = (\mathcal{F}_i)_{i=1}^m$ flags in fibres $V_{a_1}, \dots, V_{a_m}$

such that:

Near a_i V has a local trivialization in which

- $\nabla = d - A$, $\quad A = dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$
  for some $\Lambda_i \in \mathfrak{h}_i$
- $\mathcal{F}_i \cong$ standard flag $\mathcal{F}_{\theta_i}$
- Λ_i preserves $\mathcal{F}_i$ $\quad$ (i.e. $\Lambda_i \in \mathfrak{p}_{\theta_i}(\mathfrak{h}_i)$)
- $\pi(\Lambda_i) \in \mathcal{O}_i \subset \mathfrak{l}_i$ $\quad$ ($\pi : \mathfrak{p}_{\theta_i}(\mathfrak{h}_i) \twoheadrightarrow \mathfrak{l}_i$)

Thm (Biquard-B. '04 building on Hitchin, Donaldson, Corlette, Simpson, Simpson, Nakajima, Sabbah, ...)

The moduli space $\mathcal{M}_{DR}(\Sigma, \underset{\sim}{\theta}, \underset{\sim}{Q})$ of isomorphism classes of such mero. connections which are stable and parabolic degree zero is

- a hyperkähler manifold
- canonically diffeo. to a space of mero. Higgs bundles
- complete if $\underset{\sim}{\theta}, \underset{\sim}{Q}$ sufficiently generic

- Higgs fields should look like $-\frac{1}{2} dQ_i + \Lambda_i \frac{dz}{z} + \text{holom.}$ near a_i
- same 'rotation' of the weights / eigenvalues as in Simpson 1990

Simpson's table (JAMS '90) (notation & extension to other G / parahoric case, PB '10)

	Dolbeault/Higgs	DR/Connections	Betti/monod.	
weights $t_{\mathbb{R}}$	$-\tau - [-\tau]$	θ	$\phi = \theta + \tau$	
eigenvalues $t_{\mathbb{C}}$	$-\frac{1}{2}(\phi + \sigma)$	$\tau + \sigma$	$\exp(2\pi i(\tau+\sigma))$	

(eigenvalues of Γ)

$t_{\mathbb{R}}$ $\quad$ $it_{\mathbb{R}}$

(eigenvalues of Λ)

$$\mathrm{Pardeg}(V,\nabla,\mathcal{F}) = \deg(V) + \sum_{1}^{m} \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\Lambda_i + \mathrm{Tr}\,\theta_i = \sum \mathrm{Tr}\,\phi_i$$

Sufficient stability conditions

If no strictly semistable points then $\mathcal{M}$ is complete

① If (V', ∇') subconnection of (V, ∇)

$$\deg V' = \sum \operatorname{Tr} \Lambda_i' = \sum \operatorname{Tr}(\tau_i' + \sigma_i') \quad \in \mathbb{Z}$$

and $\operatorname{Tr} \tau_i' = \sum_{j \in S} (\tau_i)_{jj}$ for some $S \subset \{1, \dots, \operatorname{rk} V\}$, $\# S = \operatorname{rk} V'$

$$\operatorname{Tr} \sigma_i' = \sum_S (\sigma_i)_{jj}$$

i.e. a "subsum" of $\sum_1^m \operatorname{Tr} \tau_i + \operatorname{Tr} \sigma_i$ is in $\mathbb{Z}$

(if $(\underset{\sim}{\tau}, \underset{\sim}{\sigma})$ off of these hyperplanes then $\mathcal{M}$ complete)

Fix $G = GL_n(\mathbb{C})$

$$\underset{\text{irregular curve}}{\boldsymbol{\Sigma}} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow \begin{array}{c} \mathcal{M}_{DR}(\boldsymbol{\Sigma}) \\ \wr\!\!\| \ \text{RH irregular isomorphism} \\ \mathcal{M}_B(\boldsymbol{\Sigma}) \end{array}$$

Locally near a_i:

$$\nabla \cong d - (\underbrace{dQ_i}_{\text{irregular part specified by irregular type}} + \Lambda \frac{dz}{z} + \cdots)$$

WLOG $\Lambda \in H_i := C_g(Q_i)$

Fix $G = GL_n(\mathbb{C})$

conjugacy class (& weights)

$\mathcal{C} \subset \underset{\sim}{H} = H_1 \times \cdots H_m \quad \left(H_i = C_G(Q_i)\right)$

$\underline{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow$

irregular curve

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big|$ RH irregular isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$

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WLOG $\Lambda \in H_i := C_g(Q_i)$

Fix $G = GL_n(\mathbb{C})$

$\bullet \bullet \bullet \bullet$

Fix $G = GL_n(\mathbb{C})$

(weighted) conjugacy class

$\mathcal{C} \subset \underset{\sim}{H}$

$\Sigma \Longrightarrow \mathcal{M}(\Sigma, \mathcal{C})$

irregular curve

Hyperkahler manifold

"Wild Hitchin space"

(Biquard-B. '04)

Hitchin-Simpson

Biquard-B.

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\parallel\!\wr$ Wild non-abelian Hodge isom.

Corlette-Donaldson

Sabbah

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\parallel\!\wr$ irregular RH isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

(See e.g. survey 1203.6607 for full details)

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$ — Algebraic integrable systems (Hitchin, Nitsure, Bottacin, Markman...)

$\Big\|\Big|$ Wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big\|\Big|$ RH irregular isomorphism

$\mathcal{M}_{B}(\Sigma, \mathcal{C})$

$$\begin{array}{ccc} ⫴ & \hookrightarrow & \mathcal{M}_{Dol}(\Sigma, \mathcal{C}) \\ & & \downarrow \\ & & \mathbb{H} \end{array}$$

Fix $G = GL_n(\mathbb{C})$

$$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$$

$\Big|\Big|\Big\wr$ wild non-abelian Hodge isom.

$$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$$ — Isomonodromy systems (as Σ varies in admissible fashion)

$\Big|\Big|\Big\wr$ irregular RH isomorphism

$$\mathcal{M}_B(\Sigma, \mathcal{C})$$

$$\begin{array}{c} \underset{\sim}{\Sigma} \\ \downarrow \\ \mathbb{B} \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathcal{M}_{DR}(\Sigma_b) & \subset & \mathcal{M}_{DR/\mathbb{B}} \\ \downarrow & & \downarrow \\ b & \in & \mathbb{B} \end{array}$$

— fibre bundle with flat nonlinear connection

e.g. Painlevé equations, Schlesinger system,

JMU system, Simply-laced isomonodromy systems

Fix $G = GL_n(\mathbb{C})$

$\mathcal{M}_{Dol}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big| \wr$ Wild non-abelian Hodge isom.

$\mathcal{M}_{DR}(\Sigma, \mathcal{C})$

$\Big|\Big|\Big| \wr$ RH irregular isomorphism

$\mathcal{M}_B(\Sigma, \mathcal{C})$ —— Nonlinear braid/mapping class group actions

$$\begin{array}{c} \Sigma \\ \downarrow \\ \mathbb{B} \end{array} \Rightarrow \pi_1(\mathbb{B}, b) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

by algebraic Poisson automorphisms

"wild mapping class groups"

e.g. Braiding of Stokes data of Cecotti-Vafa / Dubrovin

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces)

E_8 E_7 E_6

D_4 $A_3 = D_3$ D_2 D_1 D_0

A_2 A_1 A_0

affine Weyl group

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces)

affine Weyl group	minimal rank of bundles	pole orders
E_8	6	1+1+1
E_7	4	1+1+1
E_6	3	1+1+1
D_4	2	1+1+1+1
$A_3 = D_3$	2	2+1+1
D_2	2	2+2
D_1	2	2+2
D_0	2	2+2
A_2	2	3+1
A_1	2	4
A_0	2	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces)

Tame $\leftarrow$ | $\rightarrow$ Wild

E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
6	4	3	2	2	2	2	2	2	2	2
1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

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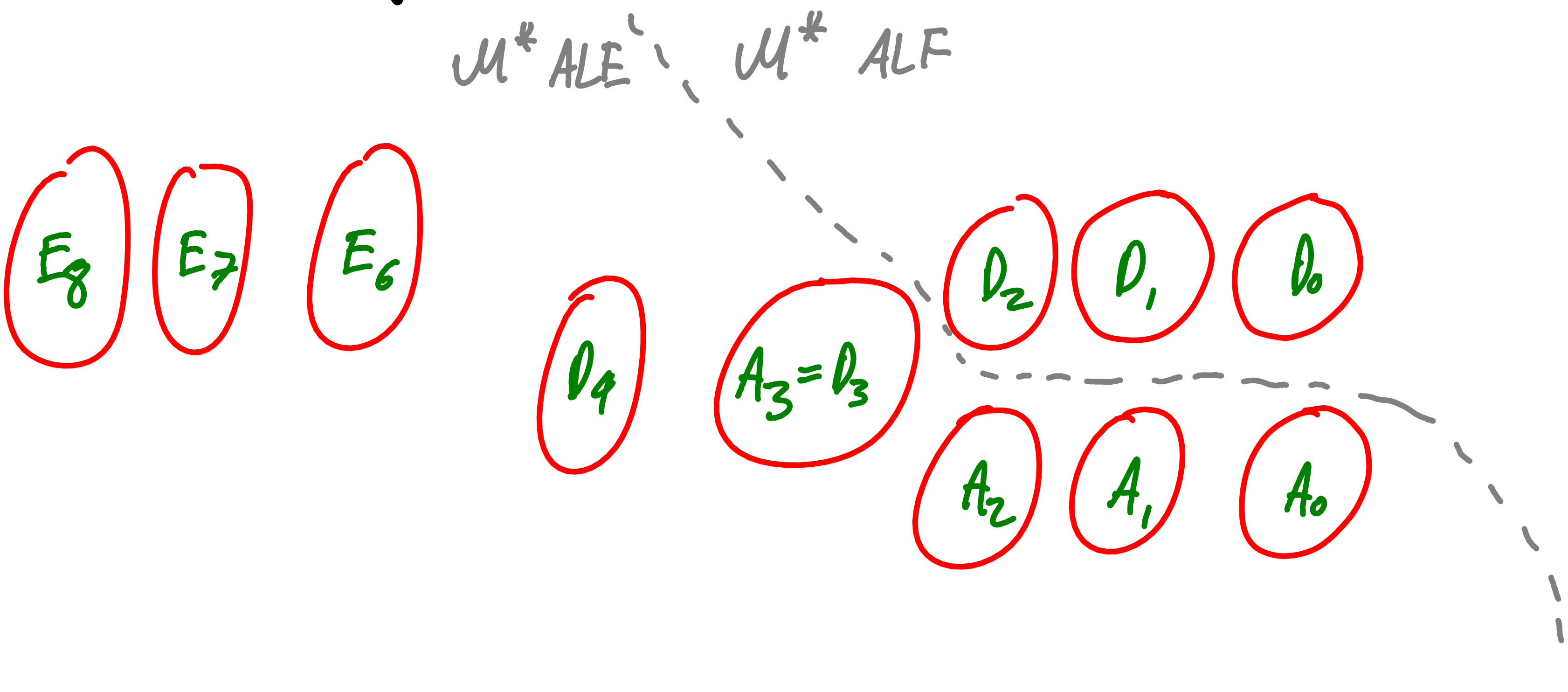


Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces)

E_8 E_7 E_6

P_3 P_3' P_3''

D_2 D_1 D_0

D_4 $A_3 = D_3$

P_6 P_5

A_2 A_1 A_0

P_4 P_2 P_1

Phase spaces for Painlevé differential equations

§1 Quiver varieties

Kronheimer, Nakajima (1990's) attached hyperkahler manifolds to graphs

graph Γ $\Rightarrow$ $\mathcal{N}(\Gamma, \lambda, d)$

ALE spaces, instantons on ALE spaces ....

§1 Quiver varieties

Kronheimer, Nakajima (1990's) attached hyperkahler manifolds to graphs

$$\text{graph } \Gamma \implies \mathcal{N}(\Gamma, \lambda, d)$$

Γ graph with nodes I, $V = \bigoplus_{i \in I} V_i$ (I graded vector space)

$d = \{d_i\}$ ($d_i = \dim V_i$) $\in \mathbb{Z}^I$, $\lambda = \{\lambda_i\} \in \mathbb{C}^I$ parameters

$$\mathrm{Rep}(\Gamma, V) = \bigoplus_{e \in \overline{\Gamma}} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \qquad t(e) \xrightarrow{\;e\;} h(e)$$

$$G = \prod_I GL(V_i) \circlearrowright \mathrm{Rep}(\Gamma, V) \quad \& \quad \boxed{\mathcal{N}(\Gamma, \lambda, d) = \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda}\, G}$$

§1 Quiver varieties

Kronheimer, Nakajima (1990's) attached hyperkahler manifolds to graphs

graph Γ $\Rightarrow$ $\mathcal{N}(\Gamma, \lambda, d) = \mathrm{Rep}(\Gamma, v) /\!/_{\lambda} G$

$\Downarrow$

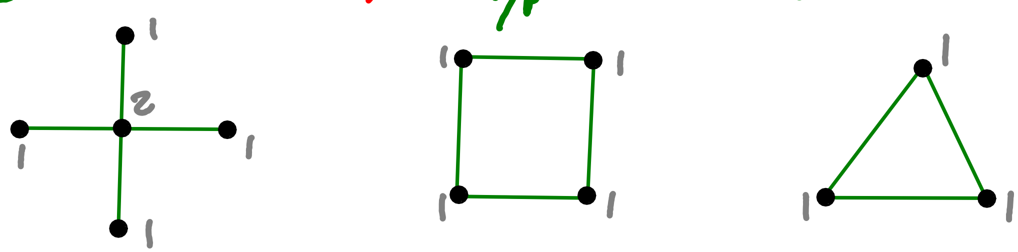
Kac-Moody algebra (Cartan matrix $C = 2 -$ adjacency matrix)

$$\mathrm{Dim}_{\mathbb{C}}(\mathcal{N}(\Gamma, \lambda, d)) = 2 - (d,d) = 2 - d \cdot Cd$$

e.g Γ affine ADE Dynkin graph, $d =$ min. imaginary root

$(d,d) = 0$, $\mathrm{Dim}_{\mathbb{C}} \mathcal{N} = 2$ $\Rightarrow$ hyperkahler four manifold (ALE)

e.g



$\sim \widehat{\mathbb{C}^2/\mathbb{Z}_3}$

§3 Compare / Relate these stories

Suppose $\Sigma = (\mathbb{P}^1, \underset{\sim}{a}, \underset{\sim}{Q})$ rational irreg. curve

Then have open subset $\mathcal{M}^*(\Sigma) \subset \mathcal{M}_{DR}(\Sigma)$

where $V \to \mathbb{P}^1$ holomorphically trivial

(moduli space of systems of linear differential operators)

Thm ('08, '11) "Modular quiver varieties"

If Γ a complete graph, or

a complete k-partite graph for any k, or

a simply-laced supernova graph

then for any $\lambda \in \mathbb{C}^I$, $d \in \mathbb{Z}^I$ $\exists$ rational Σ

such that $\mathcal{N}(\Gamma, \lambda, d) \cong \mathcal{M}^*(\Sigma, e)$ for some e

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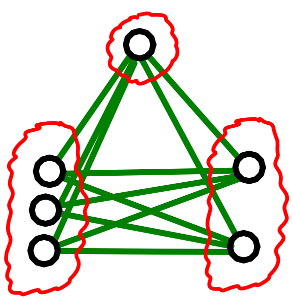
Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



$\Longleftrightarrow$ $1+2+3=6$

$\Gamma(3,2,1)$

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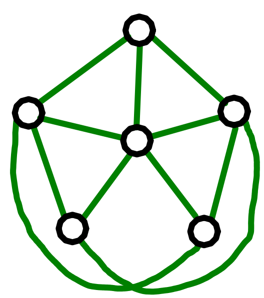

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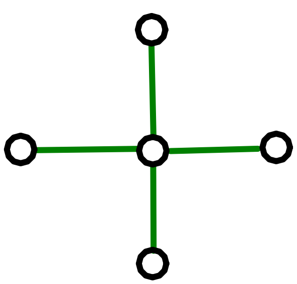
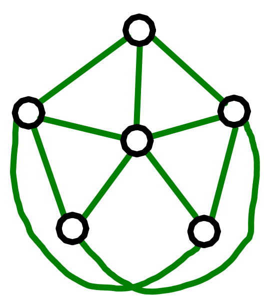
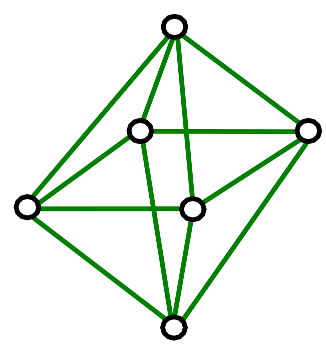
Complete k partite graphs



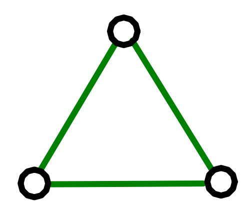
$\Gamma(1,4)$



$\Gamma(3,2,1)$



$\Gamma(2,2,2)$



$\Gamma(1,1,1)$

Thm ('08, '11) "*Modular quiver varieties*"

If Γ a complete graph, or

a complete k-partite graph for any k, or

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then for any $\lambda \in \mathbb{C}^I$, $d \in \mathbb{Z}^I$ $\exists$ rational Σ

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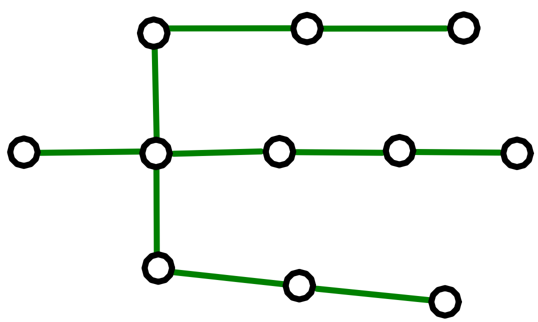
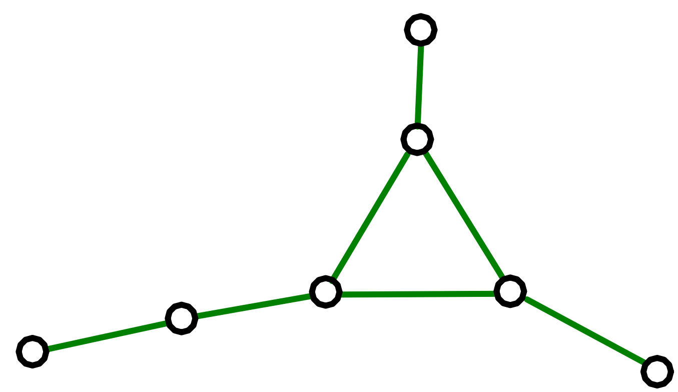
Complete k partite graphs + legs = simply-laced supernova graphs





Thm ('08, '11) "*Modular quiver varieties*"

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[Star-shaped (tame) case due to Nakajima / Crawley-Boevey]

Thm ('08, '11) "Modular quiver varieties"

- If Γ a complete graph, or
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then for any $\lambda \in \mathbb{C}^I$, $d \in \mathbb{Z}^I$ $\exists$ rational Σ such that $\mathcal{N}(\Gamma, \lambda, d) \cong \mathcal{M}^*(\Sigma, e)$ for some e

[Star-shaped (tame) case due to Nakajima / Crawley-Boevey]

- Fourier–Laplace $\Rightarrow$ reflection isom.s
  (integrable system + isomonodromy connection preserved)

"Modular quiver varieties"

Idea/example $\quad \Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$

$V = \mathbb{C}^n$, $\quad A \in \mathrm{End}(V)$ diagonal, $\quad V = \bigoplus V_i \quad$ (eigenspaces)

$A = \sum a_i \, \mathrm{Id}_{V_i} \qquad a_i \in \mathbb{C} \qquad$ (eigenvalues)

Any $\nabla \in \mathcal{M}^*(\Sigma)$ has form

$$dQ + B\,dz = (2Az + B)\,dz, \quad B \in \mathrm{End}(V)$$

Must have irreg. type dQ at ∞:

$\delta : \mathrm{End}(V) \longrightarrow \bigoplus \mathrm{End}(V_i)$

$$dQ + B\,dz \overset{G[\![z^{-1}]\!]}{\cong} dQ + \delta(B)\,dz + \Lambda \frac{dz}{z} + \cdots$$

for some ! $\Lambda \in \mathfrak{h} = \bigoplus \mathrm{End}(V_i)$

complete graph nodes $\{a_i\}$

so $\delta(B) = 0$ & $B \in \bigoplus_{i \neq j} \mathrm{Hom}(V_i, V_j) = \mathrm{Rep}(\Gamma, V)$

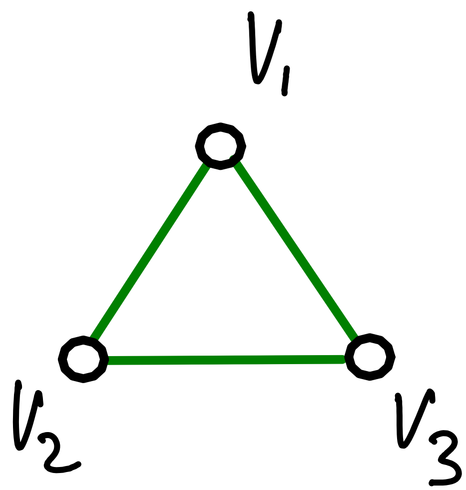
and Λ = moment map for $\mathbb{G} = \prod GL(V_i) \circlearrowright \mathrm{Rep}(\Gamma, V)$

"Modular quiver varieties"

Idea/example

$$\Sigma = (\mathbb{P}^1,\ \infty,\ Q = Az^2)$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{pmatrix} \qquad \Lambda_i \in \mathrm{End}(V_i)$$



$\dim V_i = d_i$

$\mathrm{Rank} = \sum d_i$

$$B = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \in \mathrm{End}\left(\bigoplus V_i\right)$$

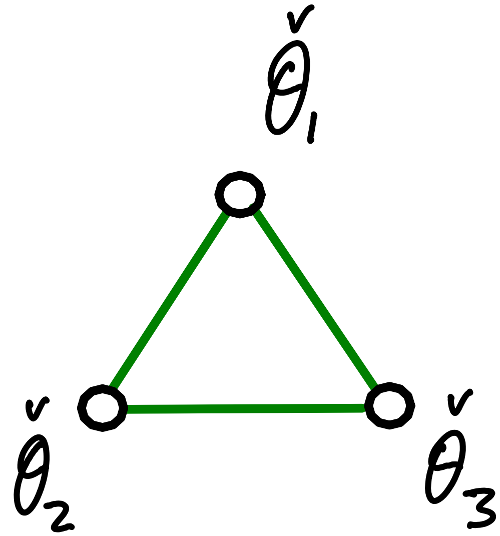
"Modular quiver varieties"

Idea/example

$$\Sigma = (\mathbb{P}^1, \infty, Q = Az^2)$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{pmatrix}$$

$\Lambda_i \in \check{\mathcal{O}}_i \subset \mathrm{End}(V_i)$ – fix orbits



$\dim V_i = d_i$

$\mathrm{Rank} = \sum d_i$

$$B = \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix} \in \mathrm{End}(\oplus V_i)$$

"Modular quiver varieties"

Idea/example

$$\Sigma = (\mathbb{P}^1,\ \infty,\ Q = Az^2)$$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{pmatrix}$$

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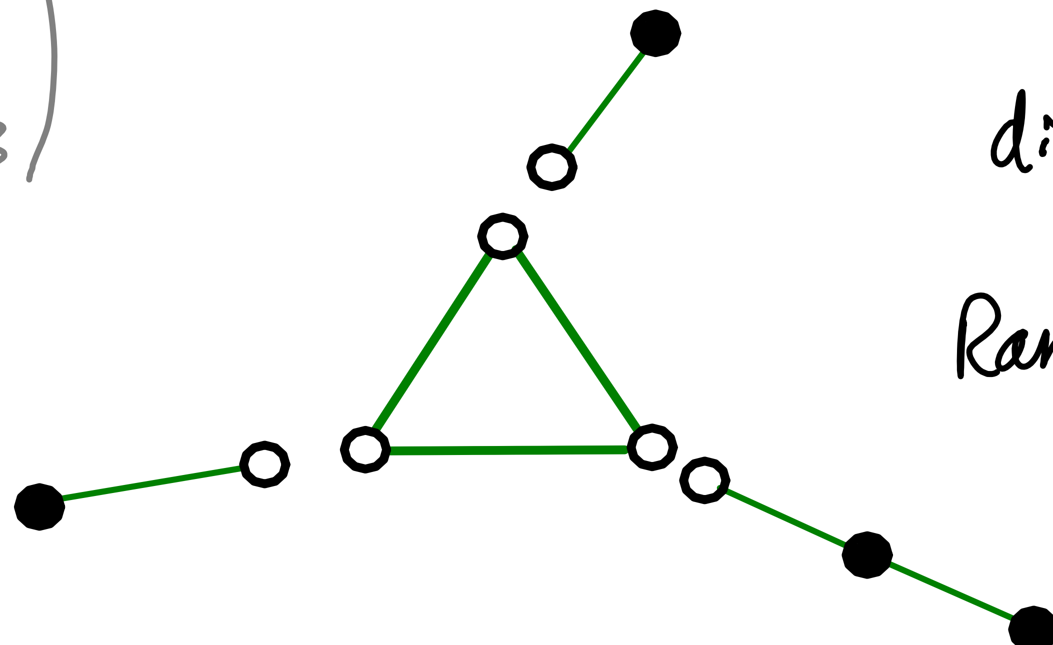
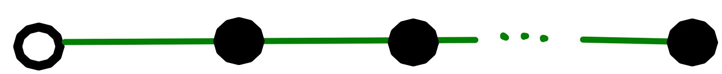


$\dim V_i = d_i$

$\mathrm{Rank} = \sum d_i$

Lemma (Kraft-Procesi, Nakajima, Crawley-Boevey)

Legs $\Leftrightarrow$ orbits

$\mathcal{O} \subset \mathrm{End}(V) \Rightarrow \mathcal{O} = \mathcal{N}($  $)$

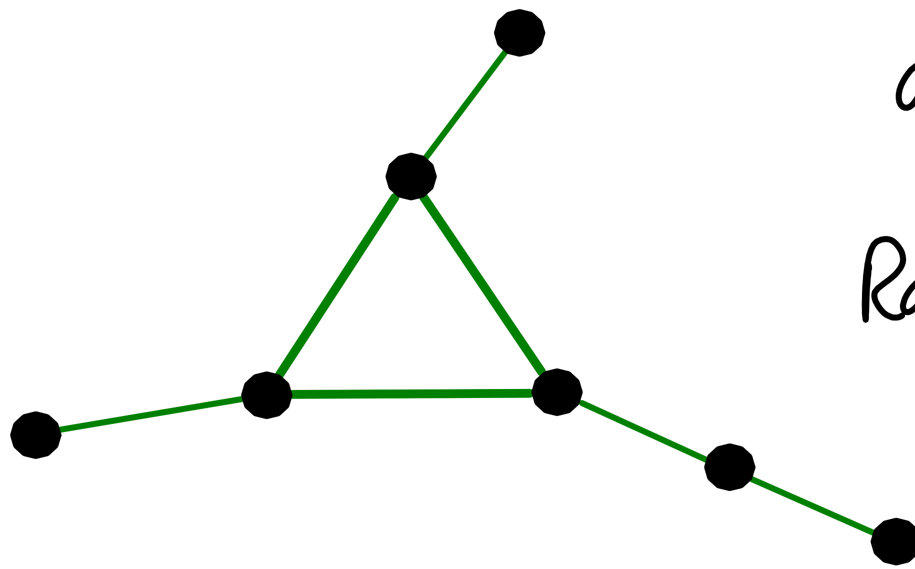
"Modular quiver varieties"

Idea/example

$\Sigma = (\mathbb{P}^1, \infty, Q = A z^2)$

$$\Lambda = \begin{pmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \Lambda_3 \end{pmatrix}$$

$\Lambda_i \in \check{\mathcal{O}}_i \subset \mathrm{End}(V_i)$ — fix orbits

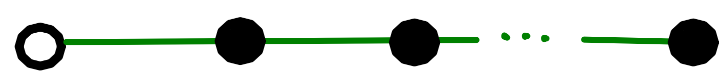


$\dim V_i = d_i$

$\mathrm{Rank} = \sum d_i$

Lemma (Kraft-Procesi, Nakajima, Crawley-Boevey)

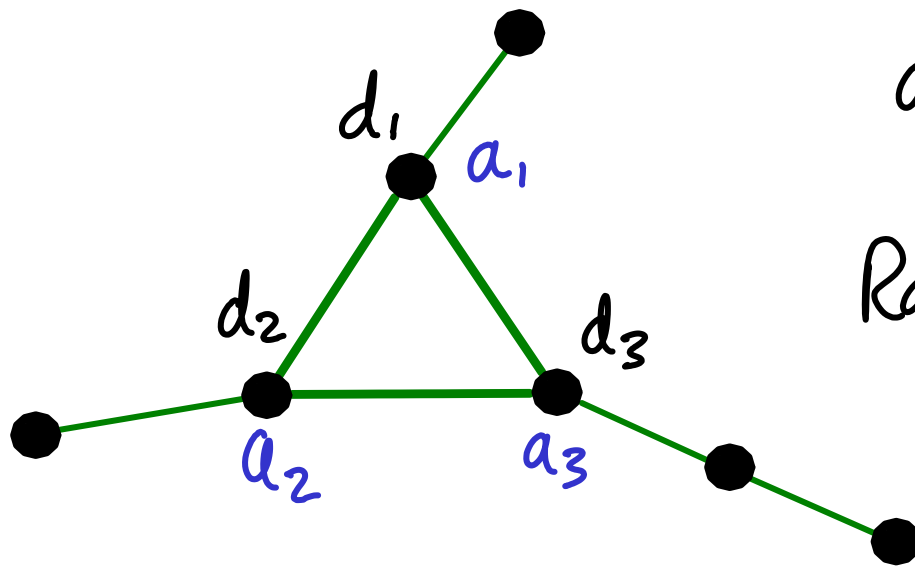
Legs $\Longleftrightarrow$ orbits

$\mathcal{O} \subset \mathrm{End}(V) \Rightarrow \mathcal{O} = \mathcal{N}($  $)$

"Modular quiver varieties"

Idea/example

$$\Sigma = \left(\mathbb{P}^1, \infty, Q = Az^2\right) \qquad \left(A = \sum a_i \, \mathrm{Id}_{V_i}\right)$$



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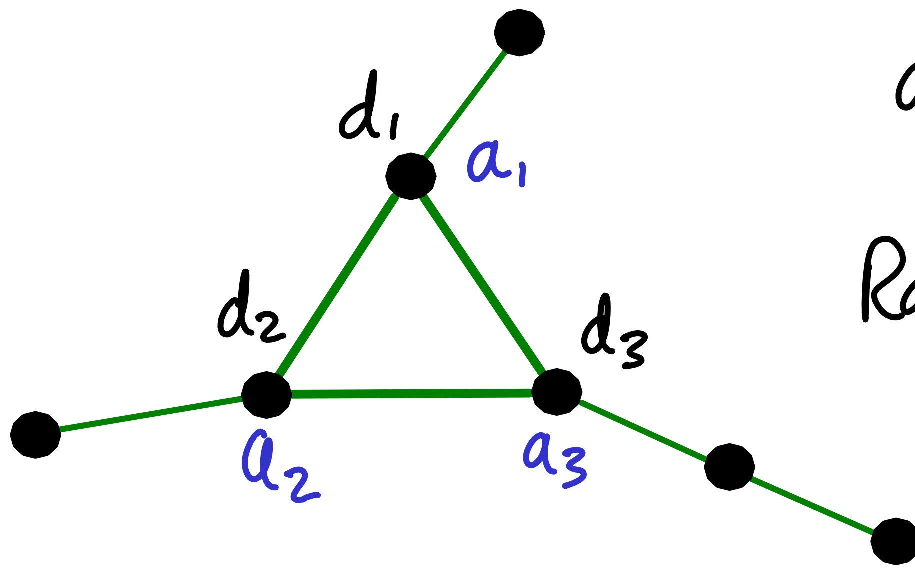

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Fourier-Laplace changes eigenvalues of A

$$a_i \longmapsto -1/a_i$$

"Modular quiver varieties"

Idea/example

Suppose $a_1 = \infty$

$$\Sigma = \left(\mathbb{P}^1, (0, \infty), (0, Az^2)\right) \quad \begin{cases} \text{Rank} = d_2 + d_3 \\ A = a_2 \mathrm{Id}_{V_2} + a_3 \mathrm{Id}_{V_3} \end{cases}$$

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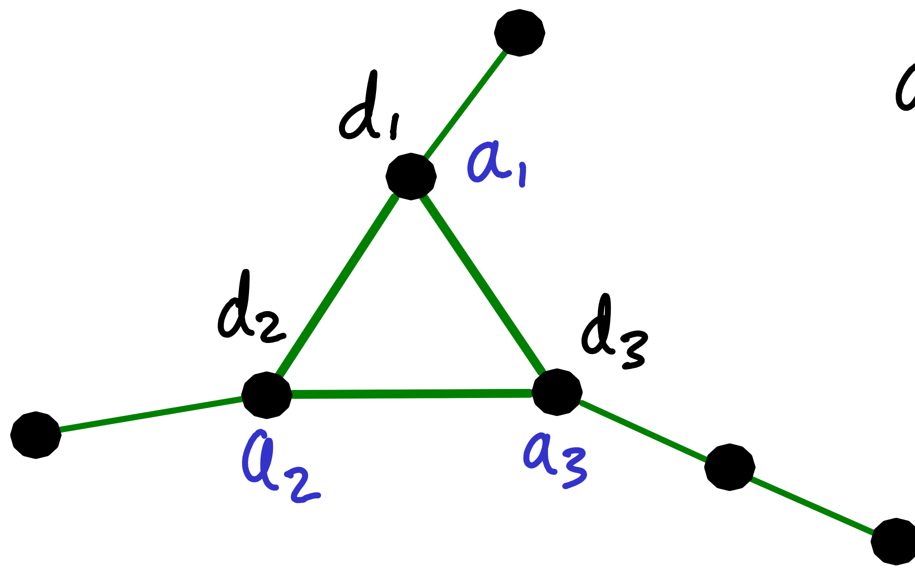

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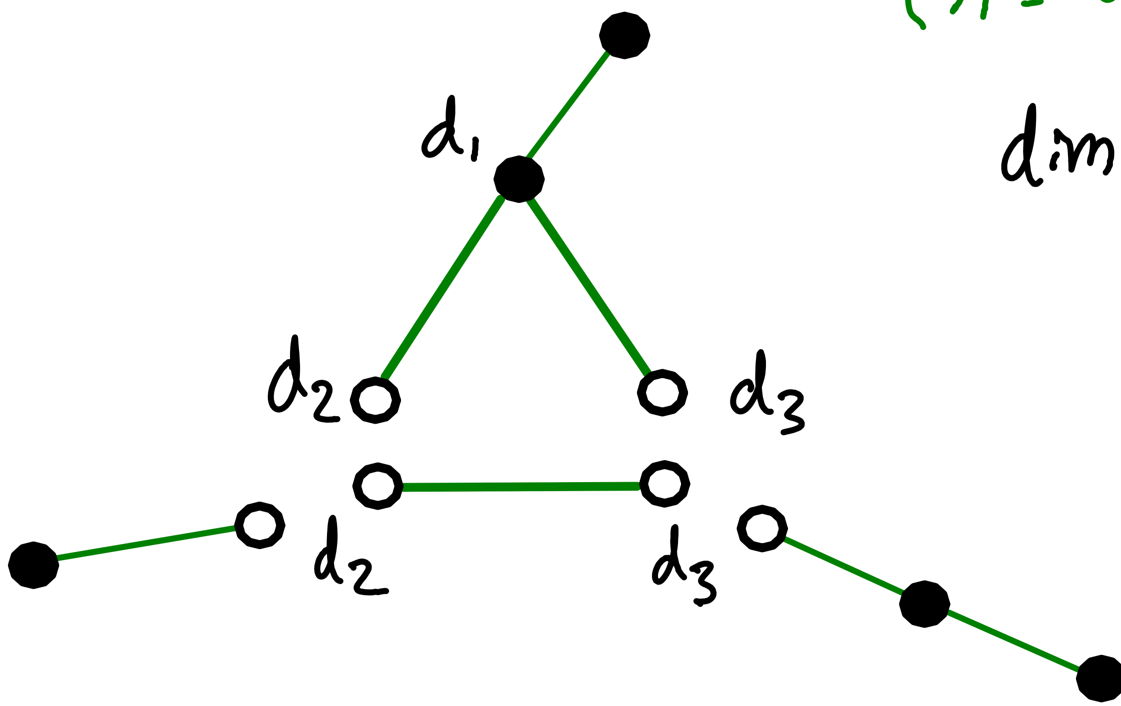

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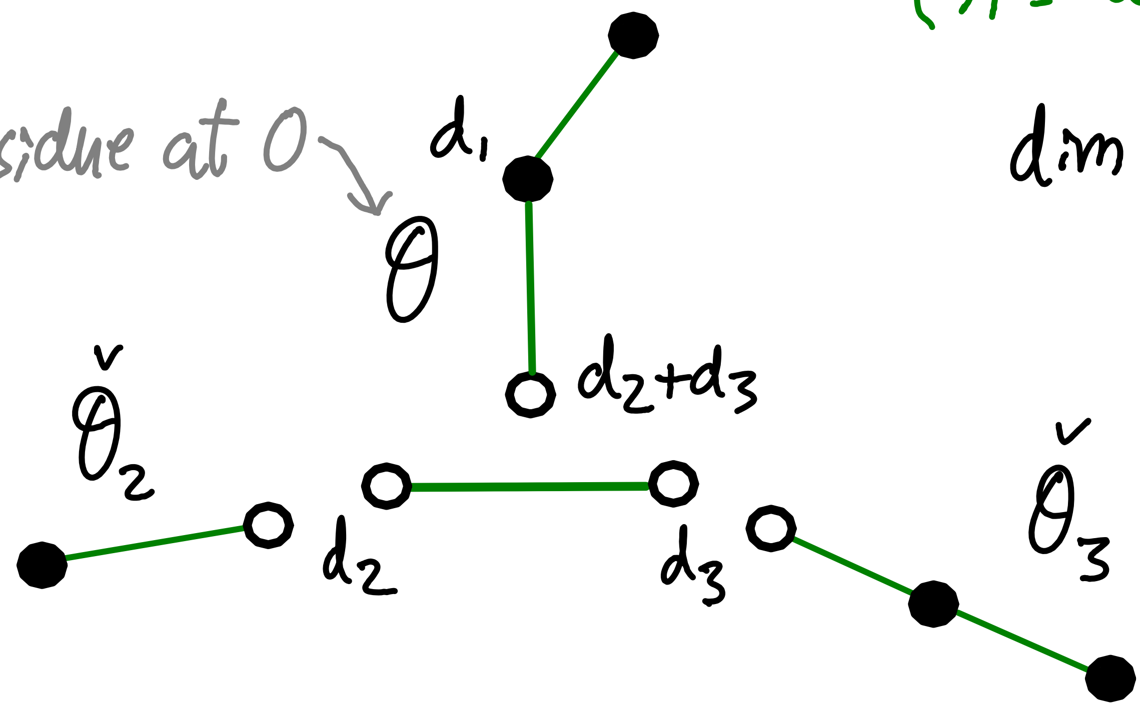
orbit of residue at 0 → θ



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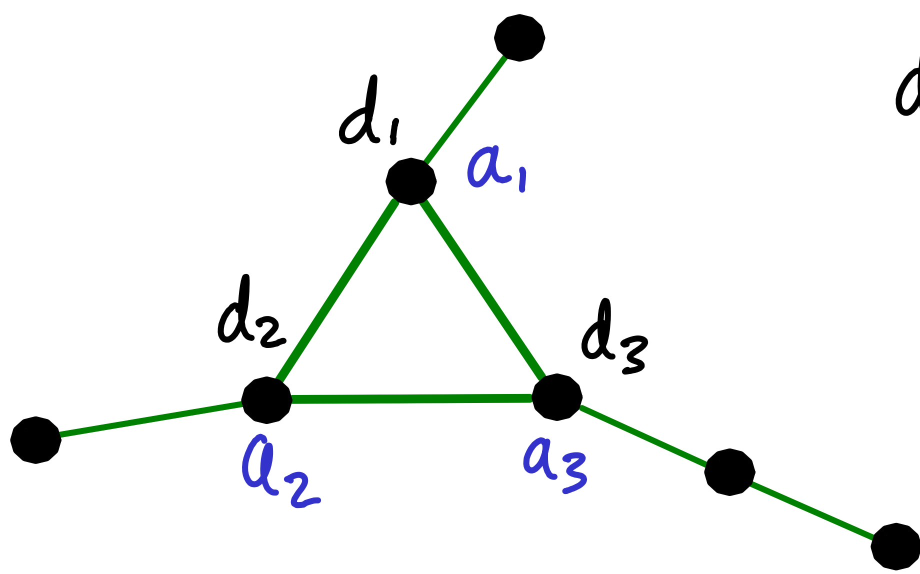
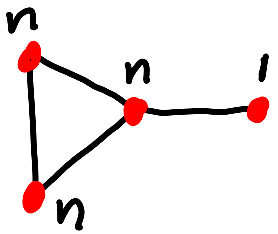


$\rightsquigarrow$ Dictionary "$k+1$ ways to read a complete k-partite graph as moduli of connections"

E.g. Higher/hyperbolic/Hilbert Painlevé systems

$\Gamma_n =$ 

$\Longrightarrow$ $hP^n_{IV} := \mathcal{M}(\Gamma_n)$ dimension $2n$

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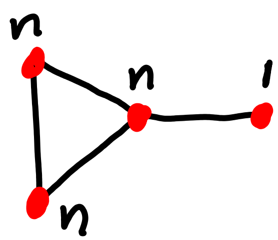
$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{IV} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$n=1 \qquad hP^1_{IV} \cong P_{IV}$ dim 2

$$\mathcal{M}^*(\Gamma_n) \underset{\text{diffeo}}{\cong} \mathrm{Hilb}^n(\mathcal{M}^*(\Gamma_1))$$

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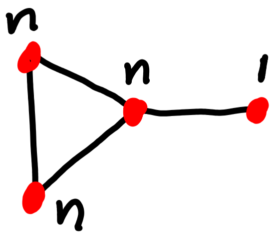
$\Gamma_n =$ 

$\Longrightarrow \quad hP_{IV}^{n} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$n=1 \qquad hP^{1}_{IV} \cong P_{IV}$ dim 2

$$\mathcal{M}^*(\Gamma_n) \underset{\text{diffeo}}{\cong} \mathrm{Hilb}^n(\mathcal{M}^*(\Gamma_1))$$

Question: $\mathcal{M}(\Gamma_n) \overset{?}{\cong} \mathrm{Hilb}^n(\mathcal{M}(\Gamma_1))$ (for generic parameters)

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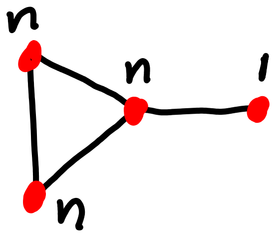
$\Gamma_n =$ 

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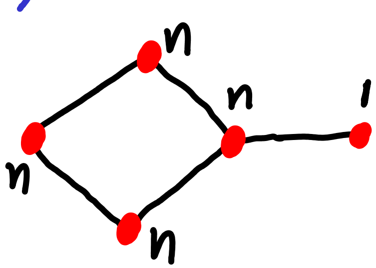
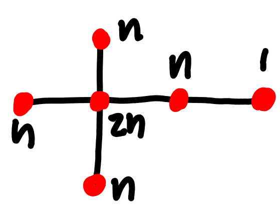
Question: $\mathcal{M}(\Gamma_n) \overset{?}{\cong} \mathrm{Hilb}^n(\mathcal{M}(\Gamma_1))$ (for generic parameters)

similarly for any 2d Hitchin system e.g.:

$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{V} := \mathcal{M}(\Gamma_n)$ dimension $2n$

$\Gamma_n =$ 

$\Longrightarrow \quad hP^n_{VI} := \mathcal{M}(\Gamma_n)$ dimension $2n$